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Instanton calculus, topological field theories and $N = 2$ Super Yang-Mills theories

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Abstract: The results obtained by Seiberg and Witten for $N = 2$ supersymmetric theories with gauge group SU(2) are in agreement with instanton computations carried out for winding numbers one and two. This suggests that the instanton saddle point saturates the non-perturbative contribution to the functional integral. A natural framework in which corrections to this approximation are absent is given by the topological field theory built out of the $N = 2$ Super Yang-Mills theory. After extending the standard construction of the Topological Yang-Mills theory to encompass the case of a non-vanishing vacuum expectation value for the scalar field, a BRST transformation is defined (as a supersymmetry plus a gauge variation), which on the instanton moduli space is the exterior derivative. We then show that each non-perturbative contribution to the $N = 2$ low-energy effective action can be written as the integral of a total derivative of a function of the instanton moduli. Only instanton configurations of zero conformal size contribute to this result.

Keywords: Topological Field Theories, Solitons Monopoles and Instantons, BRST Symmetry, Supersymmetric Effective Theories.

Contents

1. Introduction

Our understanding of the non-perturbative sector of field and string theories has greatly progressed in recent times. In [1], for the first time, the entire non-perturbative contribution to the holomorphic part of the wilsonian effective action was computed for $N = 2$ globally supersymmetric (SUSY) theories with gauge group SU(2), using

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ansätze dictated by physical intuitions. A few years later, a better understanding of non-perturbative configurations in string theory led to the conjecture that certain IIB string theory correlators on an $AdS_5 \times S^5$ background are related to Green's functions of composite operators of an $N = 4$ SU(N_c) Super Yang-Mills (SYM) theory in four dimensions in the large N_c limit [2]. Although supported by many arguments, these remarkable results remain conjectures and a clear mathematical proof seems to be out of reach at the moment. In our opinion this state of affairs is mainly due to the lack of adequate computational tools in the non-perturbative region. To the extent of our knowledge, the only way to perform computations in this regime in SUSY theories and from first principles is via multi-instanton calculus. Using this tool, many partial checks have been performed on these conjectures, both in $N = 2$ and $N = 4$ SUSY gauge theories [3]–[7]. The limits on these computations come from the exploding amount of algebraic manipulations to be performed and from the lack of an explicit parametrization of instantons of winding number greater than two [8]. In order to develop new computational tools that might allow an extension to arbitrary winding number, we revisit instanton computations for $N = 2$ in the light of the topological theory built out of $N = 2$ SYM, i.e. the so-called Topological Yang-Mills theory (TYM) [9].[1]

That the TYM might play an important rôle in instanton computations became apparent with the results of [3, 4, 5]. The agreement of these computations with the results of [1] pointed out that instantons saturate all the non-perturbative contributions to the $N = 2$ SYM low-energy effective action, and that the saddle point expansion around the classical solution does not receive any perturbative correction. This situation seems to be related to a sort of localization theorem [11]. In this respect $N = 2$ stands as an isolated case. Its effective action can be separated into a holomorphic part (which encodes the geometry of the quantum moduli space of vacua) and a non-holomorphic one. In [12] a powerful non-renormalization theorem for the former was found. Subsequently, the quantum holomorphic piece was exactly determined [1], and its non-perturbative part checked against instanton calculations [3, 4, 5]. It is this kind of contributions that we claim can also be computed from a closely related topological field theory. The story for the non-holomorphic terms is completely different. Indeed, the leading instanton contribution to the higher-derivative terms in the effective action gets perturbative corrections [13], in a way similar to what occurs in the $N = 4$ SUSY theory, in which the non-perturbative contributions to all the relevant correlators get also perturbatively corrected, as a consequence of the vanishing of the R-symmetry anomaly.

[1] This is somewhat different from previous work [10] in which the spectral curves which describe the moduli space of vacua for $N = 2$ theories with various gauge groups were put in relation with integrable systems which, in turn, are related to 2-dimensional topological field theories. The study of the relationship between this approach and the one we present here goes beyond the scope of this paper.

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From now on we will focus on the $N = 2$ case. First of all, we generalize the standard approach to TYM [9] to encompass the case in which the scalar field acquires a non-vanishing vacuum expectation value (v.e.v.), and define a BRST operator on field space, which on the moduli space $\mathcal{M}^+$ ($\mathcal{M}^-$) of (anti-)self-dual gauge connections acts as the exterior derivative. This allows us to show that the computation of the relevant Green's functions boils down to integrating differential forms on $\mathcal{M}^+$ ($\mathcal{M}^-$). More precisely, after integrating out the quantum fluctuations one is left with a theory living on the instanton moduli space. To describe this space we will make use of the ADHM construction [8].

The TYM framework allows for a better understanding of the geometry underlying the computation of correlators of observables, and casts also new light on old problems concerning instanton calculus. On one hand we will learn that the BRST operator built with a zero v.e.v. for the scalar and that obtained with a non-zero v.e.v. cannot be smoothly deformed one into the other. Thus, it does not make sense trying to match computations performed in these two different regimes. On the other hand, in the case of non-zero scalar v.e.v., the twisted formulation naturally leads to the construction of the so-called "constrained instanton" field configurations [14, 15] thus giving a firmer basis to this approach. The Ward identities associated to the scalar supersymmetry transformations show that the constrained instanton computational method actually gives the correct result for the Green's functions of physical observables. This in turn implies that, as argued before, instantons saturate the non-perturbative contribution to the relevant Green's functions, and shows how the non-renormalization theorem of [12] explicitly works in the context of instanton calculus.

It is worth remarking that in the geometrical approach outlined here, the instanton measure for the $N = 2$ SYM theory (i.e. the integration measure over the moduli, or collective coordinates) emerges in a very natural way, without resorting to any intricate zero-mode calculation as in previous approaches. In the same vein, we derive with purely algebraic methods an explicit realization of the BRST algebra on instanton moduli space. This derivation is deeply related to the construction of this space as a hyperkähler quotient [16], which we study in the last section.

In the case of non-zero scalar v.e.v., the instanton action, i.e. the $N = 2$ SYM action functional computed on the zero-modes (in our picture these are the field configurations onto which the action functional of TYM projects) can be interpreted as the commutator of the BRST charge with an appropriate function. This leads to the possibility of writing the correlators of physical observables as integrals of total differentials on $\mathcal{M}^+$. The circumstance that these Green's functions can be computed in principle on the boundary of $\mathcal{M}^+$ may greatly help in computations, since instantons on $\partial\mathcal{M}^+$ obey a kind of dilute gas approximation, as we will explain in subsection 5.3. This might lead to recursion relations of the type found in [17, 18, 19], and simplify instanton calculations. We also explore how the geometrical approach described here works in the case of vanishing v.e.v., and apply these ideas to the computation of correlators in Witten's topological field theory.

To avoid misunderstandings, we stress that the fact that certain correlators of $N=2$ SYM can be calculated using the formalism of the TYM does not mean that the former is a topological theory: $N=2$ SYM is a "physical" theory with its own degrees of freedom and a running coupling constant. In fact, TYM is formally derived from $N=2$ SYM by the twisting procedure which, in flat space, turns out to be just a variable redefinition. However, promoting the scalar supersymmetry generator present in the twisted $N=2$ algebra to a BRST charge implies great changes in the physical interpretation of the theory; some fields become ghosts and their engineering dimensions change [20]. TYM theory deserves its name topological because it is a theory with zero degrees of freedom, whose correlators can be related to topological invariants of the 4-dimensional manifold on which the theory lives. Also in SUSY gauge theories a class of position-independent correlators exists [15, 21]. One realizes that a subset of correlators of $N=2$ SYM coincides with a subset of the observables defined in TYM over $\mathbb{R}^4$: as a consequence, these Green's functions can be computed in either theory, according to one's preferences.

Summarizing, we believe this approach provides us with a natural and simplifying framework for investigating the non-perturbative dynamics of SUSY gauge theories. An abbreviated account of part of the results described here was presented in [22].

This paper is organized as follows. In subsection 2.1 we recall some basics of topological field theories with vanishing v.e.v. for the scalar field. We derive the set of identities which define a BRST operator and show that the functional integration projects the fields onto the subspace of the zero-modes of the relevant kinetic operators in the instanton background. In subsection 2.2 we generalize this discussion to the case of a non-vanishing v.e.v. and clarify the relationship between our approach and the constrained instanton computational method. In subsection 3.1, after an introduction to the ADHM construction of instantons, we write the solutions to the equations of motion derived from the TYM action. In subsection 3.2 we use the results of the previous subsection and the identities associated to the BRST symmetry to find how the BRST charge acts on the relevant quantities defined in the ADHM construction (e.g. the instanton moduli) in the absence of a v.e.v. for the scalar field, and in subsection 3.3. in the case of a non-vanishing v.e.v. We present in subsection 3.4 a purely algebraic (and independent) derivation of the BRST algebra on instanton moduli space and of the solutions to the equations of motion (which were obtained in the previous subsections). In section 4 we discuss how to compute instanton-dominated Green's functions using the formalism we have developed. It is important to understand how in our approach the instanton measure arises. This crucial issue is discussed in subsection 4.1, where we also study in detail the cases of winding number one and two. In subsection 4.2 we com-

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pute the multi-instanton action (which is non zero when the scalar field acquires a non-vanishing v.e.v.) and show that it can be written as a BRST-exact quantity. Section 5 is devoted to the calculation of $u = \langle \mathrm{Tr}\, \phi^2 \rangle$, the gauge invariant quantity which parametrizes the moduli space of quantum vacua of the $N = 2$ SYM theory (from which one can obtain the Seiberg-Witten low-energy wilsonian action using Matone's relation [17]). First, we find a general expression in our framework for the k-instanton contribution to u. Then, on one hand, in subsections 5.1 and 5.2, we compute $\langle \mathrm{Tr}\, \phi^2 \rangle$ in the bulk of $\mathcal{M}^+$ for winding numbers $k = 1, 2$; on the other hand, in subsection 5.3, using the observation of subsection 4.2, we show that the contribution to u can be written as a total derivative integrated on the moduli space of instantons. This suggests the interesting possibility of computing it directly on the boundary of $\mathcal{M}^+$; we explicitly check this in a $k = 1$ computation, getting the correct result. In section 6 we consider the case of a vanishing v.e.v. (to which our formalism also applies), and compute $\langle \mathrm{Tr}\, \phi^2 \, \mathrm{Tr}\, \phi^2 \rangle$ for winding number $k = 1$ both in the bulk and on the boundary of the instanton moduli space. Finally, in section 7 we construct the metric on the $8k$-dimensional moduli space of self-dual gauge connections for winding number $k = 2$ following the aforementioned hyperkähler quotient procedure.

2. Topological Yang-Mills theory

It is well known that, if the generators of the rotation group of $\mathbb{R}^4$ are redefined in a suitably twisted fashion, the $N = 2$ SYM theory gives rise to the TYM theory considered in [9]. A key feature of the twisted theory is the presence of a scalar fermionic symmetry Q, which is still an invariance of the theory when this is formulated on a generic (differentiable) four-manifold M. This scalar symmetry will play a major rôle, for the following reasons. First, the correlation functions of the physical observables of the theory (the cohomology classes of Q) are independent of the metric on M by virtue of the Ward identities associated to Q [9]. This also implies that these functions must be independent of the positions of the operatorial insertions.[2] Moreover, the same Ward identities entail that certain Green's functions can be computed exactly in the semiclassical limit; this is why instantons come into play. Finally, when one modifies the scalar supersymmetry charge Q to make it nilpotent, the resulting (BRST) operator acts as the exterior derivative on the anti-instanton moduli space $\mathcal{M}^-$ [23]. As we will see, functional integration reduces to integrating differential forms on $\mathcal{M}^-$ (this is what we call the localization procedure). We will later show that the Green's functions of the observables can be written as integrals of total derivatives on $\mathcal{M}^-$.

[2] That in some SUSY gauge theories there exists a class of position-independent correlators was observed in [21].

Let us first recall how the twisting operation works [9]. The global symmetry group of the $N = 2$ SUSY theory in flat space is

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_A \times \mathrm{U}(1)_R\,, \tag{2.1}$$

where the first two factors represent the euclidean Lorentz group (i.e. the rotation group of $\mathbb{R}^4$), while $\mathrm{SU}(2)_A$ is the automorphism group of the $N = 2$ supersymmetry algebra and $\mathrm{U}(1)_R$ is the usual R-symmetry. The twist consists in replacing one of the factors of the rotation group, say for definiteness $\mathrm{SU}(2)_R$, with a diagonal subgroup $\mathrm{SU}(2)'_R$ of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_A$. The symmetry group of the twisted theory is then

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)'_R \times \mathrm{U}(1)_R\,. \tag{2.2}$$

With respect to the twisted group, the SUSY charges decompose as a scalar Q, a self-dual antisymmetric tensor $Q_{\mu\nu}$ and a vector Q_μ:

$$\bar{Q}^{\dot{A}}_{\dot{\alpha}} \longrightarrow Q \oplus Q_{\mu\nu}\,, \qquad Q^{\dot{A}}_{\alpha} \longrightarrow Q_\mu\,. \tag{2.3}$$

In particular, the charge Q belongs to the $(0,0)^1$ representation of (2.2), while the charges $Q_\mu, Q_{\mu\nu}$ belong respectively to the $(1/2,1/2)^{-1}$ and $(0,1)^1$ representations.[3] In the twisted theory it is natural to redefine the fields of $N = 2$ SYM as

$$A_\mu \longrightarrow A_\mu\,, \qquad \bar{\lambda}^{\dot{A}}_{\dot{\alpha}} \longrightarrow \eta \oplus \chi_{\mu\nu}\,, \qquad \lambda^{\dot{A}}_{\alpha} \longrightarrow \psi_\mu\,, \qquad \phi \longrightarrow \phi\,. \tag{2.4}$$

The anticommuting fields $\eta, \chi_{\mu\nu}, \psi_\mu$ are respectively a scalar, a self-dual antisymmetric tensor and a vector, belonging to the $(0,0)^{-1}$, $(0,1)^{-1}$ and $(1/2,1/2)^1$ representations of the twisted group; the gauge field A_μ and the scalar field ϕ belong respectively to the $(1/2,1/2)^0$ and $(0,0)^2$ representation.

In the following we will be mainly interested in the multiplet of fields (A_μ, ψ_μ, ϕ), whose transformations under the action of Q read

$$QA_\mu = \psi_\mu\,, \qquad Q\psi_\mu = -D_\mu\phi\,, \qquad Q\phi = 0\,. \tag{2.5}$$

These equations imply that Q is nilpotent modulo gauge transformations with parameter ϕ, since

$$Q^2 A_\mu = -D_\mu\phi\,, \qquad Q^2\psi_\mu = -[\phi, \psi_\mu]\,, \qquad Q^2\phi = 0 \tag{2.6}$$

(this is analogous to what one comes across in studying the supersymmetry algebra in the Wess-Zumino gauge). (2.6) allows us to regard Q as a BRST-like charge. To this end, let us assign Q a ghost number equal to 1; this can be done by simply identifying its R-charge with the ghost number. Accordingly, the fields of the twisted $N = 2$ vector multiplet (2.4) acquire a ghost number equal to their respective R-charge. Since the canonical dimension of the BRST operator is usually taken to be zero, we redefine the canonical dimension of Q to zero. The resulting canonical dimensions and ghost numbers of the fields are summarized in table 1.

[3]With the upper index we denote the R-symmetry charge.

At this point we need to distinguish the case in which the scalar v.e.v. vanishes from that in which it is non vanishing. In the next subsection we will focus on the former situation; the latter requires a more detailed discussion, and will be studied separately in subsection 2.2.

Fields	A	ψ	χ	η	ϕ	$\bar{\phi}$
dimension	1	1	2	2	0	2
ghost #	0	1	-1	-1	2	-2

Table 1: Canonical dimensions and ghost numbers of the TYM fields.

2.1 Case I: Zero vacuum expectation value for the scalar field

As (2.6) shows, Q is not nilpotent. A strictly nilpotent BRST charge can be obtained from Q by introducing a generalized BRST operator s including both the gauge symmetry and the scalar supersymmetry of the theory [24],

$$s = s_g + Q\,. \tag{2.7}$$

s_g is the usual BRST operator associated to the gauge symmetry,

$$s_g A = -Dc\,, \qquad s_g\psi = -[c,\psi]\,, \qquad s_g\phi = -[c,\phi]\,, \qquad s_g c = -\frac{1}{2}[c,c]\,, \tag{2.8}$$

the ghost number and canonical dimension of the ghost field c being respectively one and zero. The 1-form $A = A_\mu dx^\mu$ is the gauge connection, with curvature $F = 1/2 F_{\mu\nu} dx^\mu dx^\nu = dA + AA$; $\psi = \psi_\mu dx^\mu$ is an anticommuting 1-form, and D is the covariant exterior derivative on the manifold M. The action of Q on the ghost field c is obtained by requiring that $s^2 = 0$, and turns out to be simply

$$Qc = \phi\,. \tag{2.9}$$

(2.5), (2.8) and (2.9) thus lead to the following BRST identities [24]:

$$\begin{aligned} sA &= \psi - Dc\,, \qquad & s\psi &= -[c,\psi] - D\phi\,, \\ s\phi &= -[c,\phi]\,, \qquad & sc &= -\frac{1}{2}[c,c] + \phi\,. \end{aligned} \tag{2.10}$$

The algebra (2.10) can be read as the definition and the Bianchi identities for the curvature

$$\widehat{F} = F + \psi + \phi \tag{2.11}$$

of the connection

$$\widehat{A} = A + c \tag{2.12}$$

of the universal bundle $P \times \mathcal{A}/\mathcal{G}$, where $P, \mathcal{A}, \mathcal{G}$ are respectively the principal bundle over M, the space of connections and the group of gauge transformations. The exterior derivative on the manifold $M \times \mathcal{A}/\mathcal{G}$ is given by [24]

$$\widehat{d} = d + s\,. \tag{2.13}$$

Notice that from the last of (2.10) we learn that the scalar field ϕ can be seen as the curvature of the connection c.

We now come to define the observables of the TYM theory; these are given by the elements of the equivariant cohomology of s [28], which satisfy the descent equations

$$\begin{aligned}
s\frac{1}{2}\,\mathrm{Tr}\,F^2 &= -d\,\mathrm{Tr}\,F\psi\,,\\
s\,\mathrm{Tr}\,F\psi &= -d\,\mathrm{Tr}\left(\phi F + \frac{1}{2}\psi^2\right),\\
s\,\mathrm{Tr}\left(\phi F + \frac{1}{2}\psi^2\right) &= -d\,\mathrm{Tr}\,\phi\psi\,,\\
s\,\mathrm{Tr}\,\phi\psi &= -\frac{1}{2}d\,\mathrm{Tr}\,\phi^2\,,\\
s\frac{1}{2}\,\mathrm{Tr}\,\phi^2 &= 0\,.
\end{aligned} \tag{2.14}$$

(2.14) allows one to build local functions of the fields which are BRST invariant modulo d-exact terms; the simplest example of a physical observable is the gauge invariant polynomial $\mathrm{Tr}\,\phi^2$, as the last of (2.14) shows. We will see in section 3 that this geometrical approach provides us with an operative tool which allows us to compute the Green's functions of observables starting only from the knowledge of the universal connection (2.12), in particular without solving any equation of motion.

As shown in [24], a TYM action can be interpreted as a pure gauge-fixing term,

$$S_{\mathrm{TYM}} = 2\int d^4x\;s\,\mathrm{Tr}\,\Psi\,, \tag{2.15}$$

where the gauge-fixing fermion is chosen to be

$$\Psi = \chi^{\mu\nu}F^+_{\mu\nu} - D^\mu\bar{\phi}\psi_\mu + \bar{c}\partial^\mu A_\mu\,, \tag{2.16}$$

and

$$F^+_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right) \tag{2.17}$$

is the self-dual component of the field strength $F_{\mu\nu}$. The anti-fields $\chi_{\mu\nu}$, $\bar{\phi}$ and $\bar{c}$ transform under the BRST symmetry as

$$s\chi_{\mu\nu} = B_{\mu\nu}\,,\qquad s\bar{\phi} = \eta\,,\qquad s\bar{c} = b\,, \tag{2.18}$$

while the Lagrange multipliers $B_{\mu\nu}$, η and b as

$$sB_{\mu\nu} = 0\,,\qquad s\eta = 0\,,\qquad sb = 0\,. \tag{2.19}$$

The anti-field $\bar{c}$ has ghost number -1 and dimension 2, while the Lagrange multipliers $(B_{\mu\nu},\, b)$ have ghost number 0 and dimension 2. Acting with the BRST operator s in (2.15), we obtain the following explicit form for the TYM action:

$$\begin{aligned} S_{\mathrm{TYM}} = 2\int d^4x\, \mathrm{Tr}\Big[& B^{\mu\nu}F^+_{\mu\nu} - \chi^{\mu\nu}(D_{[\mu}\psi_{\nu]})^+ + \eta D^\mu\psi_\mu - \\ & - \bar{\phi}(D^2\phi - [\psi^\mu, \psi_\mu]) + b\partial^\mu A_\mu + \\ & + \chi^{\mu\nu}[c, F^+_{\mu\nu}] - \bar{\phi}[c, D^\mu\psi_\mu] - \bar{c}s(\partial^\mu A_\mu)\Big]\,, \end{aligned} \tag{2.20}$$

where

$$(D_{[\mu}\psi_{\nu]})^+ = \frac{1}{4}\Big(D_\mu\psi_\nu - D_\nu\psi_\mu + \epsilon_{\mu\nu\rho\sigma}D^\rho\psi^\sigma\Big) \tag{2.21}$$

is the self-dual component of the tensor $D_{[\mu}\psi_{\nu]}$. (2.20) is obtained integrating by parts the term in $\bar{\phi}$ of (2.16); the corresponding surface term vanishes, since in this subsection we limit ourselves to study the case in which all the fields have trivial boundary conditions.

The main property of the action (2.20) is that *it localizes the fields in the algebra (2.10) onto certain sections of the universal bundle.* In particular, functional integration over the fields $B^{\mu\nu}$ and $(\chi^{\mu\nu},\, \eta)$ in the first line of (2.20) leads respectively to

$$F^+_{\mu\nu} = 0\,, \tag{2.22}$$

which implies that A is an anti-self-dual gauge connection, and

$$\big(D_{[\mu}\psi_{\nu]}\big)^+ = 0\,, \qquad D^\mu\psi_\mu = 0\,, \tag{2.23}$$

which entails that ψ is an element of the tangent bundle $T_A\mathcal{M}^-$. In turn, functional integration over the anti-field $\bar{\phi}$ leads to the equation

$$D^2\phi = [\psi^\mu, \psi_\mu] \tag{2.24}$$

for the field ϕ, while integration on the Lagrange multiplier b imposes the usual transversality condition

$$\partial^\mu A_\mu = 0\,. \tag{2.25}$$

By plugging the expression for ψ deduced from the first equation in (2.10) into the transversality condition $D^\mu\psi_\mu = 0$, we get

$$D^2c = -D^\mu(sA_\mu)\,, \tag{2.26}$$

which determines the ghost field c. Two observations are in order. First, let us remark that the first equation in (2.10) is an old acquaintance [25, 26]. It is in fact very well known that differentiating the gauge connection with respect to collective coordinates (sA) fails to give a transverse zero-mode (ψ). To ensure the correct gauge condition, the addition of a gauge transformation (Dc) is needed: (2.26) just determines this gauge transformation. A suitable framework for multi-instanton calculations is given by the ADHM construction. It is interesting to discover that this construction, together with the TYM formalism, allows one to write the universal connection (2.12) (and consequently the ghost c and the gauge transformation Dc) in a very natural and straightforward way [29]. We will focus on this aspect in section 3. Finally, the terms in the last line of (2.20) vanish due to the conditions (2.22), (2.23) and (2.25).

Summarizing, we have seen that after functional integration on the anti-fields and the Lagrange multipliers, we are left with an integration on the space of anti-self-dual gauge connections $\mathcal{M}^-$ and its tangent bundle $T_A\mathcal{M}^-$, with a functional measure equal to 1, since the action $S_{\rm TYM}$ vanishes on the field subspace identified by the (zero-mode) equations (2.22)–(2.26).

Notice that in this approach the functional integral is performed exactly, since the gauge-fixing fermion (2.16) is linear in the antifields, and there are no perturbative corrections. It is important to remark that the action obtained by twisting the $N=2$ SYM theory (i.e. the action of Witten's topological field theory [9]) actually differs from (2.20) by some extra terms, which spoil the linearity of $S_{\rm TYM}$ in the anti-fields. However, as we will show below, they are BRST-exact terms corresponding to a continuous deformation of the gauge-fixing

$$S_{N=2} = S_{\rm TYM} + s\mathcal{V}\,; \tag{2.27}$$

the v.e.v. of an s-closed operator $\mathcal{O}$, i.e. such that $s\mathcal{O}=0$, is controlled by the Ward identity [9] (in the following $[\delta\varphi]$ is shorthand for the integration measure)

$$\begin{aligned}\langle\mathcal{O}\rangle_{S+s\mathcal{V}} &\equiv \int[\delta\varphi]e^{-(S+s\mathcal{V})}\mathcal{O} = \int[\delta\varphi]e^{-S}\mathcal{O}(1-s\mathcal{V}+\cdots)\\ &= \langle\mathcal{O}\rangle_S - \langle\mathcal{O}s\mathcal{V}\rangle_S+\cdots = \langle\mathcal{O}\rangle_S - \langle s(\mathcal{O}\mathcal{V})\rangle_S+\cdots\\ &= \langle\mathcal{O}\rangle_S\,,\end{aligned} \tag{2.28}$$

the last equality following from the fact that the v.e.v. of an s-exact operator $\mathcal{P}=s\mathcal{Q}$ vanishes if $\mathcal{Q}$ is globally defined.[4] This means that the action (2.20) and the twisted $N=2$ SYM action are completely equivalent, in the sense that *the Green's functions of s-closed operators can be computed using any one of them obtaining the same result*. We now show that the lagrangian obtained by twisting the $N=2$ SYM theory can be derived by modifying the gauge-fixing fermion (2.16), thus proving that the twisted version of $N=2$ SYM action introduced in [9] and the TYM action (2.20) differ only by BRST-exact terms [24]. To this end, let us consider the modified gauge-fixing fermion

$$\Psi^{(\alpha)} = \chi^{\mu\nu}\left(F^+_{\mu\nu}-\frac{\alpha}{2}B_{\mu\nu}\right) - D^\mu\bar\phi\psi_\mu + \bar c\partial^\mu A_\mu\,, \tag{2.29}$$

[4] In this respect we remark that $\mathrm{Tr}\,\phi^2$ is not globally defined. This fact plays an important rôle in breaking $N=2$ SUSY into $N=1$ [30].

where α is a gauge-fixing parameter; upon functional integration over the Lagrange multiplier $B_{\mu\nu}$, one gets

$$
\begin{aligned}
S^{(\alpha)}_{\rm TYM} = 2\int d^4x \,{\rm Tr}\Big[\frac{1}{2\alpha}F^{+\mu\nu}F^+_{\mu\nu} - \chi^{\mu\nu}(D_{[\mu}\psi_{\nu]})^+ + \eta D^\mu\psi_\mu - \\
- \bar\phi(D^2\phi - [\psi^\mu,\psi_\mu]) + b\partial^\mu A_\mu + \\
+ \chi^{\mu\nu}[c, F^+_{\mu\nu}] - \bar\phi[c, D^\mu\psi_\mu] - \bar c s(\partial^\mu A_\mu)\Big] .
\end{aligned}
\tag{2.30}
$$

This functional leads to the same equations of motion of the $N=2$ SYM theory.

Three observations are in order. First, notice that the partition function (or more generally the Green's functions) defined by (2.30) has to be studied using the usual saddle point techniques due to the presence of the kinetic term for the gauge field. As discussed in [9], supersymmetry ensures the cancellation of bosonic and fermionic determinants arising when integrating out the non-zero modes [27] and the resulting integration is over $\mathcal{M}^-$, which is the same result obtained from the use of the action (2.20). Second, the renormalization group invariant scale (which multiplies the Green's functions computed in the conventional supersymmetric theory) does not appear. In fact the divergent term of the one-loop effective action is proportional to $(F^+)^2$ which is obviously vanishing on $\mathcal{M}^-$ [31].

Last, and most importantly, let us observe that the equivalence between $S^{(\alpha)}_{\rm TYM}$, eq. (2.30), and $S_{\rm TYM}$, eq. (2.20), in the computation of correlators of observables is not surprising, since they cannot depend on the choice of the gauge parameter α; therefore nothing prevents us from choosing directly $\alpha = 0$. As we will discuss in section 2.2, in the presence of a non-trivial v.e.v. for the scalar field, the equivalence of (2.20) to the $N=2$ SYM action (in the sense previously specified) makes the configurations of the constrained instanton method emerge from functional integration, without any approximation procedure.

Let us now sketch the geometrical interpretation of the instanton calculus suggested by the topological formulation. To begin with, the first equation in (2.10) together with (2.23) imply that, as announced, the BRST operator s has an intriguing explicit realization on the moduli space as the exterior derivative [23]. Furthermore, once the universal gauge connection (2.12) is given, the other field configurations ψ, ϕ are in turn immediately determined as components of the universal curvature (2.11), as it will be worked out in detail in section 3. Topological correlators are then built up as differential forms on the moduli space, where the form degree of the fields equals their ghost number. For example, for winding number $k=1$ the top form on the (8-dimensional) instanton moduli space is given by ${\rm Tr}\,\phi^2(x_1)\,{\rm Tr}\,\phi^2(x_2)$. We will explicitly compute the corresponding Green's function in section 6, both with a bulk calculation and with a calculation on the boundary of the instanton moduli space, finding the same result.

2.2 Case II: Non-zero vacuum expectation value for the scalar field

In this subsection we will extend the construction of the TYM to encompass the presence of a non-vanishing v.e.v. for the scalar field. To this end, observe first that a non-zero v.e.v. for ϕ,

$$\lim_{|x|\to\infty} \phi = v\frac{\sigma_3}{2i}\,, \tag{2.31}$$

implies the existence of a (non-zero) central charge Z in the SUSY algebra. Then the operator defined in (2.7) is no longer nilpotent; instead, it closes on a U(1) central charge transformation

$$\begin{aligned}(s_g+Q)^2 A &= ZA \equiv -D\phi_Z\,,\\ (s_g+Q)^2 \psi &= Z\psi \equiv -[\phi_Z,\psi]\,,\\ (s_g+Q)^2 \phi_Z &= Z\phi_Z \equiv 0\,,\end{aligned} \tag{2.32}$$

where the scalar field ϕ_Z plays the rôle of a gauge parameter and satisfies the equation

$$D^2\phi_Z = 0\,, \qquad \lim_{|x|\to\infty} \phi_Z = v\frac{\sigma_3}{2i}\,. \tag{2.33}$$

Notice that from (2.32) it follows that the central charge Z has ghost number two and canonical dimension zero. We also remind that the scalar charge Q commutes with s_g, while Z commutes with all the other charges by definition.

From (2.32) it follows that, in order to ensure the nilpotency property, the operator (2.7) has to be properly extended by including the central charge. We then define an extended BRST operator [32] as

$$s = s_g + Q - \lambda Z + \frac{\partial}{\partial\lambda}\,, \tag{2.34}$$

where λ is a fermionic parameter with ghost number -1 and canonical dimension zero, such that λZ has the usual quantum numbers of a BRST operator. It is easy to see that the last term of (2.34) is needed to ensure the nilpotency of s; in fact, by using (2.32) and the property $\lambda^2 = 0$, we get

$$s^2 = (s_g+Q)^2 - \frac{\partial}{\partial\lambda}(\lambda Z) = 0\,. \tag{2.35}$$

If we define the ghost field

$$\Lambda \equiv \lambda\phi_Z\,, \tag{2.36}$$

with the transformation

$$s\Lambda = \phi_Z - [c,\Lambda]\,, \tag{2.37}$$

we can finally write the resulting BRST algebra as

$$\begin{aligned} sA &= \psi - D(c+\Lambda)\,, & s\psi &= -[c+\Lambda,\psi] - D\phi\,,\\ s\phi &= -[c+\Lambda,\phi]\,, & s(c+\Lambda) &= -\frac{1}{2}[c+\Lambda,c+\Lambda] + \phi\,. \end{aligned} \tag{2.38}$$

Notice that now the curvature ϕ of the ghost $c+\Lambda$ is decomposed as the sum

$$\phi = \phi_Z + \phi_g \,, \tag{2.39}$$

where ϕ_g is related to the usual gauge transformations

$$sc = \phi_g - \frac{1}{2}[c,c] \,, \qquad \lim_{|x|\to\infty} \phi_g = 0 \,. \tag{2.40}$$

(2.36) entails that the field Λ, which has ghost number one and dimension zero, can be seen as the ghost related to the central charge symmetry. This fact can be analyzed in more detail by considering eqs. (2.23) for the ψ field: in the presence of a non-zero central charge, they no longer have a unique solution, since any field configuration of the form $\psi - D\tilde{\Lambda}$ satisfies the same equations on the anti-instanton background $F^+_{\mu\nu} = 0$

$$\left(D_{[\mu}\psi_{\nu]}\right)^+ - \left[F^+_{\mu\nu}, \tilde{\Lambda}\right] = \left(D_{[\mu}\psi_{\nu]}\right)^+ = 0 \,, \qquad D^\mu\psi_\mu - D^2\tilde{\Lambda} = D^\mu\psi_\mu = 0 \,, \tag{2.41}$$

provided that

$$D^2\tilde{\Lambda} = 0 \,, \qquad \lim_{|x|\to\infty} \tilde{\Lambda} \neq 0 \,; \tag{2.42}$$

according to (2.33), this identifies $\tilde{\Lambda}$ as the parameter of a central charge transformation. This degeneracy will be removed once the boundary conditions for the ghost $\tilde{\Lambda}$ are fixed. Notice that if the central charge were zero, the equation $D^2\tilde{\Lambda} = 0$ would have trivial boundary conditions; its solution would be $\tilde{\Lambda} = 0$, and the degeneracy would disappear, as expected. In our case instead, from (2.33) and (2.36) it follows that Λ is a solution of (2.42) satisfying the boundary condition

$$\lim_{|x|\to\infty} \Lambda = \lambda v \frac{\sigma_3}{2i} \tag{2.43}$$

induced by the asymptotic behavior of the scalar field ϕ. The new BRST algebra could then be derived from (2.10) by just shifting the ghost c to $c+\Lambda$; c is related to the usual gauge transformations, whereas Λ takes into account the new U(1) transformations generated by the central charge.

Let us now evaluate the action (2.15). This can be done by noticing that, according to the algebra (2.38), its explicit form can be obtained simply by substituting the ghost c in (2.20) with its shifted version $c+\Lambda$. Unlike the case studied in the previous subsection, the action gets now a contribution from integrating by parts the term $s\,\mathrm{Tr}(D^\mu\bar{\phi}\psi_\mu)$ which does not vanish due to the non-trivial boundary conditions (2.31). Explicitly

$$s\int d^4x\, 2\,\mathrm{Tr}\left[\left(D^\mu\bar{\phi}\right)\psi_\mu\right] = s\int d^4x\, 2\,\partial^\mu\,\mathrm{Tr}\left(\bar{\phi}\psi_\mu\right) - s\int d^4x\, 2\;\mathrm{Tr}\left(\bar{\phi}D^\mu\psi_\mu\right). \tag{2.44}$$

We then get

$$
\begin{aligned}
S_{\mathrm{TYM}} = 2\int d^4x\, \mathrm{Tr}\Big[& B^{\mu\nu}F^+_{\mu\nu} - \chi^{\mu\nu}(D_{[\mu}\psi_{\nu]})^+ + \eta D^\mu\psi_\mu - \\
& - \bar\phi\Big(D^2\phi - [\psi^\mu,\psi_\mu]\Big) + b\partial^\mu A_\mu + \\
& + \chi^{\mu\nu}[c+\Lambda, F^+_{\mu\nu}] - \bar\phi[c+\Lambda, D^\mu\psi_\mu] - \bar c s(\partial^\mu A_\mu)\Big] - \\
- 2s\int d^4x\, 2\,\mathrm{Tr}(\bar\phi D^\mu\psi_\mu) & + 2\int d^4x\, \partial^\mu s\, \mathrm{Tr}(\bar\phi\psi_\mu)\,.
\end{aligned} \tag{2.45}
$$

The functional integration over the anti-fields and the Lagrange multipliers goes as in subsection 2.1, and leads to the same set of (zero-mode) equations

$$F^+_{\mu\nu} = 0\,, \tag{2.46}$$

$$(D_{[\mu}\psi_{\nu]})^+ = 0\,, \tag{2.47}$$

$$D^\mu\psi_\mu = 0\,, \tag{2.48}$$

$$D^2\phi = [\psi^\mu,\psi_\mu]\,; \tag{2.49}$$

the key difference with respect to section 2.2 is that now the scalar field ϕ has non-trivial boundary conditions as per (2.31). It is worth remarking that the (zero-mode subspace) configurations dictated by (2.46)–(2.49), together with the boundary condition (2.31), are exactly those which are exploited in the context of the constrained instanton method [14, 15] as *approximate* solutions to the saddle point equations. Instead, as we have explained, in our approach such field configurations naturally come into play after functional integrating the anti-fields and the Lagrange multipliers; no approximation is involved. The Ward identity (2.28) of the previous subsection applies also here, implying the equivalence of the action (2.45) to that of the $N = 2$ SYM theory in computing the Green's functions of the physical observables. This explains why the constrained instanton method gives the correct result for the calculation of these correlators.

Once the connection $\widehat{A} = A + c + \Lambda$ is known, the BRST identities (2.38) provide us with the field configurations of A, ψ, ϕ which solve the equations of motion (2.46)–(2.49). This possibility, and the circumstance that the BRST operator acts on instanton moduli space as the exterior derivative conspire to make it possible to explicitly work out (in the ADHM formalism) the aforementioned field configurations without solving their equations of motion. In the next section we will first guess the ADHM expression for $\widehat{A}$, and then show how the procedure outlined here works.

As in the zero v.e.v. case, functional integration is performed exactly, and we are left with an integration over $\mathcal{M}^-$ and $T_A\mathcal{M}^-$. No perturbative renormalization of the physical correlators calculated with TYM is allowed, in agreement with the non-renormalization theorem of [12]. Green's functions are built up as differential forms on the moduli space also in this case. However, the functional measure is now crucially different from 1, since the action computed on the zero-mode subspace gets a non-vanishing contribution $S_{\rm inst}$ from the last term of (2.45), which reads

$$S_{\rm inst} = 2 \int d^4x \, \partial^\mu s \, {\rm Tr}(\bar{\phi}\psi_\mu) \, . \tag{2.50}$$

This in turn implies that $\exp(-S_{\rm inst})$ acts as a generating functional for differential forms on the moduli space. This gives rise to non-trivial correlation functions which take contribution from topological sectors of *any* winding number k. The most interesting example is the v.e.v. of the gauge invariant, s-exact operator ${\rm Tr}\,\phi^2$, i.e. $u(v) = \langle {\rm Tr}\,\phi^2 \rangle$, which plays a prominent rôle in the context of the Seiberg-Witten model. In section 5 we will focus on this particular Green's function. First, in (5.9) we will give the general expression for the contribution to $u(v)$ coming from the topological sector of winding number k; furthermore in subsections 5.1 and 5.2 we will perform the computation for instanton number equal to one and two, respectively. Last, in subsection 5.3 we will illustrate the possibility of computing $u(v)$ with a calculation on the boundary of instanton moduli space. To support this idea we will work out the $k = 1$ computation explicitly.

3. The BRST algebra on instanton moduli space

When restricted to configurations which obey the equations of motion dictated by the TYM action, the BRST algebra gets realized on instanton moduli space. In the following we will construct this realization explicitly. To this end we will start by briefly recalling some basic elements of the ADHM construction of instantons, which provides us with a parametrization of this moduli space. This description is given in terms of a redundant set of parameters; we will then focus on its reparametrization symmetries, which will play a major rôle in the following. Our first goal will be the construction of the BRST algebra on instanton moduli space starting from the knowledge of the solutions to (2.22), (2.23), (2.24) and (2.26), for a generic winding number, which were found in [8, 33, 4]. In our set-up we will also need a new ingredient, i.e. the solution to (2.26) for the ghost field c, which we will obtain *ex novo*.

However, a completely different path could be followed: indeed, we will show that it is possible to construct the algebra *directly* on instanton moduli space, in particular without solving any field equation. This is an important remark, since in this way the construction of the algebra acquires a geometrical meaning and stands on its own. This approach is further developed in section 7, where we show its close relationship with the hyperkähler quotient construction of the instanton moduli space.

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3.1 Construction of the solutions to the equation of motion in the ADHM formalism

In section 2 we saw that the TYM action localizes the fields relevant to the BRST transformations in (2.10) onto a set of configurations dictated by a system of coupled differential equations. Here we will review how to construct explicit solutions to these equations of motion.

To begin with, recall that gauge fields A are projected onto instanton configurations.[5] As it is well known, self-dual SU(2) connections on S^4 can be put into one to one correspondence with holomorphic vector bundles of rank 2 over $\mathbb{CP}^3$ admitting a reduction of the structure group to its compact real form. The ADHM construction [8] is an algorithm which gives all these holomorphic bundles and consequently all SU(2) connections on S^4 (this S^4 should be thought of as the conformal compactification of $\mathbb{R}^4$. For the construction of instantons on $\mathbb{R}^4$, see for example [34]).

The construction is purely algebraic and we find it more convenient to use quaternionic notations. The points, x, of the quaternionic space $\mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$ can be conveniently represented in the form $x = x^\mu \sigma_\mu$, with $\sigma_\mu = (i\sigma_c, \mathbb{1}_{2\times 2})$, $c = 1, 2, 3$. The σ_c's are the usual Pauli matrices, and $\mathbb{1}_{2\times 2}$ is the 2-dimensional identity matrix. The conjugate of x is $x^\dagger = x^\mu \bar{\sigma}_\mu$. A quaternion is said to be real if it is proportional to $\mathbb{1}_{2\times 2}$ and imaginary if it has vanishing real part.

The prescription to find an instanton of winding number k is the following: introduce a $(k+1)\times k$ quaternionic matrix linear in x

$$\Delta = a + bx \,, \tag{3.1}$$

where a has the generic form

$$a = \begin{pmatrix} w_1 & \cdots & w_k \\ & & \\ & a' & \\ & & \end{pmatrix}; \tag{3.2}$$

a' is a $k \times k$ quaternionic matrix. The (anti-hermitean) gauge connection is then written as

$$A = U^\dagger dU \,, \tag{3.3}$$

where U is a $(k+1)\times 1$ matrix of quaternions providing an orthonormal frame of $\mathrm{Ker}\,\Delta^\dagger$, i.e.

$$\Delta^\dagger U = 0 \,, \tag{3.4}$$

$$U^\dagger U = \mathbb{1}_{2\times 2} \,. \tag{3.5}$$

[5] In the previous section we adopted the standard convention in topological field theories of taking the gauge curvature to be anti-self-dual. Unfortunately the literature on instanton calculus adopts the opposite convention (self-dual), to which we will conform from now on.

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The constraint (3.5) ensures that A is an element of the Lie algebra of the SU(2) gauge group. The condition of self-duality

$$ {}^*F = F \tag{3.6} $$

on the field strength of (3.3) is imposed by restricting the matrix Δ to obey

$$ \Delta^\dagger \Delta = (\Delta^\dagger \Delta)^T \,, \tag{3.7} $$

where the superscript T stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). (3.7) in turn implies $\Delta^\dagger\Delta = f^{-1} \otimes \mathbb{1}_{2\times 2}$, where f is an invertible hermitean $k \times k$ matrix (of real numbers). From (3.3), the field strength of the gauge field can be computed and it is

$$ F = U^\dagger d\Delta f d\Delta^\dagger U \,. \tag{3.8} $$

From this one can derive the following remarkable expressions for $\mathrm{Tr}(FF)$ [35, 36] (see also [37]):[6]

$$ \mathrm{Tr}(FF) = -\frac{1}{2} \Box\Box \ln \det \Delta^\dagger \Delta d^4x \tag{3.9} $$

$$ = d\,\mathrm{Tr}\left[PdD(dD)^\dagger D(dD)^\dagger + \frac{1}{3}(D^\dagger dD)(D^\dagger dD)(D^\dagger dD)\right] , \tag{3.10} $$

where

$$ P = UU^\dagger = 1 - \Delta f \Delta^\dagger \tag{3.11} $$

is the projector on the kernel of $\Delta^\dagger$, and according to [35] the columns of Δ, which are independent, have been orthonormalized and collected into a matrix we have called D.

Gauge transformations are implemented in this formalism as right multiplication of U by a unitary (possibly x-dependent) quaternion. Moreover, A is invariant under reparametrizations of the ADHM data as follows:

$$ \Delta \longrightarrow Q\Delta R \,, \tag{3.12} $$

with $Q \in \mathrm{Sp}(k+1), R \in \mathrm{GL}(k, \mathbb{R})$. It is straightforward to see that (3.12) preserves the bosonic constraint (3.7). These symmetries can be used to simplify the expressions of a and b. Exploiting this fact, in the following we will choose the matrix b to be

$$ b = -\begin{pmatrix} 0_{1\times k} \\ \mathbb{1}_{k\times k} \end{pmatrix} . \tag{3.13} $$

[6]The conventions used in this paper imply that the Pontryagin index is given by $-1/(8\pi^2)\int \mathrm{Tr}(FF)$, and it is positive (negative) on (anti-)instanton configurations.

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Choosing the canonical form (3.13) for b, the bosonic constraint (3.7) becomes

$$a' = a'^{T} \,, \tag{3.14}$$

$$a^{\dagger}a = (a^{\dagger}a)^{T} \,. \tag{3.15}$$

Moreover, in this case there still exist left-over $\mathrm{O}(k) \times \mathrm{SU}(2)$ reparametrizations of the form (3.12), where now $R \in \mathrm{O}(k)$,

$$Q = \begin{pmatrix} q & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R^{T} & \\ 0 & & & \end{pmatrix} , \tag{3.16}$$

and $q \in \mathrm{SU}(2)$. These transformations act non-trivially on the matrix a and leave b invariant. After imposing the constraint (3.7), the number of independent degrees of freedom contained in Δ (that is the number of independent collective coordinates that the ADHM formalism uses to describe an instanton of winding number k) is $8k + k(k-1)/2$; modding out the $\mathrm{O}(k) \times \mathrm{SU}(2)$ reparametrization transformations, we would get $8k - 3$ truly independent degrees of freedom. However (3.4) and (3.5) do not determine $U_0/|U_0|$, where U_0 is the first component of U; this adds three extra degrees of freedom, so that in conclusion we end up with a moduli space of dimension $8k$ (the instanton moduli space $\mathcal{M}^{+}$). It is easy to convince oneself that the arbitrariness in $U_0/|U_0|$ can be traded for the SU(2) reparametrizations; in other words, one can forget to mod out the SU(2) factor of the reparametrization group $\mathrm{O}(k) \times \mathrm{SU}(2)$ but fix the phase of the quaternion U_0 (setting for example $U_0 = |U_0| \mathbb{1}_{2\times 2}$). This is what we will actually do in the following.

We now focus our attention on the other fields involved in the BRST algebra (2.10). To begin with, the TYM action projects the anti-commuting 1-form ψ_μ onto the solutions to

$$^{*}\left(D_{[\mu}\psi_{\nu]}\right) = D_{[\mu}\psi_{\nu]} \,, \qquad D_\mu \psi_\mu = 0 \,, \tag{3.17}$$

where D is the covariant derivative in the instanton background, eq. (3.3). The solution to (3.17) can be written as [38]

$$\psi = U^{\dagger}\mathcal{M}f\left(d\Delta^{\dagger}\right)U + U^{\dagger}(d\Delta)f\mathcal{M}^{\dagger}U \,, \tag{3.18}$$

where $\mathcal{M}$ is a $(k+1) \times k$ matrix of quaternions, whose elements are Grassmann variables; moreover, in order for (3.18) to satisfy (3.17), $\mathcal{M}$ must obey the constraint

$$\Delta^{\dagger}\mathcal{M} = \left(\Delta^{\dagger}\mathcal{M}\right)^{T} \,. \tag{3.19}$$

(3.17) tell us that the ψ zero-modes are the tangent vectors to the instanton moduli space $\mathcal{M}^{+}$; as it is well known, the number of independent zero-modes is $8k$ (the dimension of $\mathcal{M}^+$), and we would like to see how this is implemented in the formalism of the ADHM construction. To this end, note that $\mathcal{M}$ has $k(k+1)$ quaternionic elements ($4k(k+1)$ real degrees of freedom) which are subject to the $4k(k-1)$ constraints given by (3.19). The number of independent $\mathcal{M}$'s satisfying (3.19) is thus $8k$, as desired.

If we work in the gauge in which b has the canonical form (3.13), then (3.19) can be conveniently elaborated as follows. We put $\mathcal{M}$ in a form which parallels the one for a in (3.2), i.e.

$$\mathcal{M} = \begin{pmatrix} \mu_1 & \cdots & \mu_k \\ & & \\ & \mathcal{M}' & \\ & & \end{pmatrix}, \tag{3.20}$$

$\mathcal{M}'$ being a $k \times k$ quaternionic matrix. Plugging (3.2), (3.13) and (3.20) into (3.19) we get

$$\mathcal{M}' = \mathcal{M}'^T, \tag{3.21}$$

$$a^\dagger \mathcal{M} = (a^\dagger \mathcal{M})^T. \tag{3.22}$$

When Δ is transformed according to (3.12), the $\mathcal{M}$'s must also be reparametrized in such a way to keep the constraint (3.19) unchanged. This implies that the $\mathcal{M}$'s undergo the same formal reparametrization of Δ, that is

$$\mathcal{M} \longrightarrow Q\mathcal{M}R. \tag{3.23}$$

We now turn to the scalar field configuration. This is dictated by (2.24), which should be supplemented by some boundary condition at infinity. Without loss of generality, we will set

$$\lim_{|x|\to\infty} \phi = \mathcal{A}_{00} = \frac{v\sigma^3}{2i}, \tag{3.24}$$

where $v \in \mathbb{C}$. The solution to (2.24) and (3.24) was found in [4] and reads

$$\phi = U^\dagger \mathcal{M} f \mathcal{M}^\dagger U + U^\dagger \mathcal{A} U, \tag{3.25}$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{A}' & \\ 0 & & & \end{pmatrix}. \tag{3.26}$$

Here $\mathcal{A}'$ is a $k \times k$ real antisymmetric matrix, and the condition for (3.25) to satisfy (2.24) is

$$\Delta^\dagger \mathcal{A} \Delta - \left(\Delta^\dagger \mathcal{A} \Delta\right)^T = -\Lambda_f, \tag{3.27}$$

where we set

$$\Lambda_f = \mathcal{M}^\dagger \mathcal{M} - \left(\mathcal{M}^\dagger \mathcal{M}\right)^T; \tag{3.28}$$

note that $\Lambda_f = -\Lambda_f^T \propto \mathbb{1}_{2\times 2}$. Hereafter, by $\mathcal{A}\Delta$ we intend the x-independent $(k+1)\times k$ matrix

$$\mathcal{A}\Delta = \begin{pmatrix} \mathcal{A}_{00}w_1 - w_m\mathcal{A}'_{m1} & \cdots & \mathcal{A}_{00}w_k - w_m\mathcal{A}'_{mk} \\ & [\mathcal{A}', a'] & \end{pmatrix} . \tag{3.29}$$

(3.29) is obtained by exploiting the fact that, in the calculations of the observables, expressions like $\mathcal{A}\Delta$ are always multiplied from the left by $U^\dagger$; therefore, recalling that $U^\dagger a = -U^\dagger bx$, we can eliminate the x-dependence in $\mathcal{A}\Delta$. Using (3.2) and (3.29), the x-dependence disappears also from the l.h.s. of (3.27), which becomes

$$\Delta^\dagger\mathcal{A}\Delta - \left(\Delta^\dagger\mathcal{A}\Delta\right)^T \equiv L\cdot\mathcal{A}' + \Lambda_b(\mathcal{A}_{00})\,; \tag{3.30}$$

according to [4], the action of L on $k\times k$ matrices Ω' is given by

$$L\cdot\Omega' = -\frac{1}{2}\{\Omega', W\} + \frac{1}{2}\,\mathrm{Tr}\left([\bar{a}', \Omega']a' - \bar{a}'[a', \Omega']\right), \tag{3.31}$$

where $W_{kl} = \bar{w}_k w_l + \bar{w}_l w_k$, and

$$[\Lambda_b]_{ij}(\Omega_0) = \bar{w}_i\Omega_0 w_j - \bar{w}_j\Omega_0 w_i\,. \tag{3.32}$$

Note that $[\Lambda_b]_{ij}(\Omega_0)$ are c-numbers when $\Omega_0^\dagger = -\Omega_0$. (3.27) can be now more compactly written as

$$L\cdot\mathcal{A}' = -\Lambda_b(\mathcal{A}_{00}) - \Lambda_f\,. \tag{3.33}$$

The structure of (3.33) suggests setting

$$\mathcal{A}' = \mathcal{A}'_b + \mathcal{A}'_f\,, \tag{3.34}$$

where

$$L\cdot\mathcal{A}'_b = -\Lambda_b(\mathcal{A}_{00})\,, \tag{3.35}$$
$$L\cdot\mathcal{A}'_f = -\Lambda_f\,. \tag{3.36}$$

This decomposition is useful since the solution ϕ_{hom} to the homogeneous equation

$$D^2\phi = 0 \tag{3.37}$$

with the non-trivial boundary condition (3.24) has the form

$$\phi_{\mathrm{hom}} = U^\dagger\mathcal{A}_b U\,, \tag{3.38}$$

where we set

$$\mathcal{A}_b = \begin{pmatrix} \mathcal{A}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{A}'_b & \\ 0 & & & \end{pmatrix}; \tag{3.39}$$

using (3.30) and (3.35) can be written as

$$\Delta^\dagger \mathcal{A}_b \Delta - \left(\Delta^\dagger \mathcal{A}_b \Delta\right)^T = 0\,, \tag{3.40}$$

which is the homogeneous equation associated to (3.27). Moreover, L is a generally invertible operator acting on $k\times k$ matrices [4]. As a consequence, $\mathcal{A}'_b \neq 0$ if and only if $\mathcal{A}_{00} \neq 0$. This is because non-trivial solutions to the homogeneous equation (3.37) exist only when non-trivial boundary conditions on ϕ are imposed. On the other hand, the solution $\phi_{\rm inh}$ to (2.24) supplemented by trivial boundary conditions

$$\lim_{|x|\to\infty} \phi_{\rm inh} = 0\,, \tag{3.41}$$

reads as

$$\phi_{\rm inh} = U^\dagger \mathcal{M} f \mathcal{M}^\dagger U + U^\dagger \mathcal{A}_f U\,, \tag{3.42}$$

with $\mathcal{A}_f$ given by

$$\mathcal{A}_f = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{A}'_f & \\ 0 & & & \end{pmatrix}. \tag{3.43}$$

As before, the reparametrization invariance (3.12) induces a transformation on the matrix $\mathcal{A}$, which can be found by requiring that the new matrix still satisfies (3.27) when Δ is replaced by its transformed expression; to this end one must have

$$\mathcal{A} \longrightarrow Q\mathcal{A}Q^\dagger\,. \tag{3.44}$$

The last field relevant to our discussion is the ghost field c, which in principle should be determined by solving (2.26). However, the definition for the universal connection $\widehat{A}$ given in (2.12), the expression (2.13) for $\widehat{d}$ together with the explicit form (3.3) for A suggest a simple guess for its ADHM expression; we write

$$c = U^\dagger (s + \mathcal{C}) U\,, \tag{3.45}$$

where $\mathcal{C}$ is the connection associated with the reparametrizations of the ADHM construction, which are shown in (3.12). Therefore, under these symmetries it transforms as

$$\mathcal{C} \longrightarrow Q(\mathcal{C} + s)Q^\dagger\,. \tag{3.46}$$

In section 2 we observed that the first equation in (2.10) together with (2.23) imply that the BRST operator s has an explicit realization on instanton moduli space as the exterior derivative. Since we are describing this space in terms of a redundant parametrization, every expression should be covariant with respect to the reparametrization symmetry group. This implies that ordinary derivatives on the instanton moduli space (which is described by the redundant set of $8k + k(k-1)/2$

ADHM collective coordinates) have to be replaced by covariant ones, and s by its covariant counterpart

$$\mathcal{S} = s + \mathcal{C}\,, \tag{3.47}$$

which is exactly what appears in (3.45). The criterion to fix $\mathcal{C}$ is clear: one has simply to plug (3.45) into (2.26), and solve for $\mathcal{C}$. In the next section we will illustrate an alternative (and quicker) way to construct $\mathcal{C}$. At this point we must alert the reader that the situation in which $\mathcal{A}_{00} \neq 0$ requires a more detailed analysis. As we will discuss in subsection 3.3, in this case the correct guess for $\mathcal{C}$ is

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}' & \\ 0 & & & \end{pmatrix}, \tag{3.48}$$

where $\mathcal{C}' = -(\mathcal{C}')^T$ and $\mathcal{C}_{00}$ is non vanishing when $\mathcal{A}_{00} \neq 0$. For the moment, let us only observe that (at least when $\mathcal{A}_{00} = 0$), $\mathcal{C}$ is from its very definition a moduli-dependent $(k+1) \times (k+1)$ matrix antisymmetric in its lowest $k \times k$ block and zero elsewhere.

In summary, we are left with four ADHM matrices:

1. Δ, which collects the ADHM data of the instanton configuration ($8k + k(k-1)/2$ degrees of freedom),
2. $\mathcal{M}$, which parametrizes the tangent vectors to the instanton moduli space ($8k$ degrees of freedom),
3. $\mathcal{A}$, which is the solution to (3.27), and
4. $\mathcal{C}$, which is the matrix in (3.48).

Under the action of the group of reparametrization of the ADHM construction, Δ and $\mathcal{M}$ transform in the fundamental representation, whereas $\mathcal{C}$ transforms as a connection and $\mathcal{A}$ as the curvature of a connection. We want to warn the reader that only Δ and $\mathcal{C}$ will emerge as independent quantities. Once they are given, all the other quantities (i.e. $\mathcal{M}$ and $\mathcal{A}$) will be completely determined, as we will show in the next subsection.

3.2 The BRST algebra in the ADHM formalism: the zero vacuum expectation value case

As we have seen, the TYM action projects the field A onto the solutions of the self-duality equations (3.6), and the anti-commuting 1-form ψ onto the tangent vectors to the instanton moduli space (the solutions to (3.17)); moreover, the scalar field ϕ must satisfy (2.24), possibly with its boundary condition (3.24), and the ghost field c

satisfies (2.26), which is induced by the transversality condition of ψ in the instanton background. In the last section we used the ADHM formalism to write the solutions to these coupled equations, that we collect here for the sake of clarity:

$$\begin{aligned} A &= U^\dagger dU\,, \qquad c = U^\dagger (s+\mathcal{C})U\,, \\ \psi &= U^\dagger \mathcal{M} f (d\Delta)^\dagger U + U^\dagger (d\Delta) f \mathcal{M}^\dagger U\,, \\ \phi &= U^\dagger \mathcal{M} f \mathcal{M}^\dagger U + U^\dagger \mathcal{A} U\,. \end{aligned} \tag{3.49}$$

The BRST transformations of these fields are written in (2.10). If we now plug (3.49) into (2.10), we end with a set of equations which will provide us with explicit expressions for the variations $s\Delta$, $s\mathcal{M}$, $s\mathcal{C}$, $s\mathcal{A}$ in terms of Δ, $\mathcal{M}$, $\mathcal{C}$, $\mathcal{A}$. At the same time we will also show how to determine the explicit form of $\mathcal{C}$.

A preliminary ingredient which is necessary for this computation is the knowledge of sU; we would like to express it in terms of $s\Delta$, otherwise we would be forced to solve the highly non-trivial set of algebraic equations (3.4) and (3.5) for U. The following trick is then useful. Perform the BRST variation of (3.4),

$$(s\Delta)^\dagger U + \Delta^\dagger sU = 0\,; \tag{3.50}$$

this can be read as an equation for $s\Delta$, whose solution is[7]

$$sU = -\Delta f (s\Delta)^\dagger U + U(U^\dagger sU)\,. \tag{3.51}$$

We are now in a position to start computing sA by varying the first equation of (3.49). This way we get

$$sA = U^\dagger \Big[s\Delta f (d\Delta)^\dagger + d\Delta f (s\Delta)^\dagger \Big] U - \big[D, U^\dagger sU \big]\,, \tag{3.52}$$

where, for a generic 1-form K, we put $[D,K] = dK + AK + KA$. Here and in the following we repeatedly use the fact that

$$\Delta^\dagger dU = -(d\Delta)^\dagger U\,, \tag{3.53}$$

which is a consequence of (3.4). We now substitute the explicit expressions found for ψ and c into the r.h.s. of the first of (2.10), thus getting

$$\psi - Dc = U^\dagger \Big(\mathcal{M} f d\Delta^\dagger + d\Delta f \mathcal{M}^\dagger \Big) U - \big[D, U^\dagger sU \big] - \big[D, U^\dagger \mathcal{C} U \big]\,. \tag{3.54}$$

If we equate the r.h.s. of (3.54) to the r.h.s. of (3.52) we obtain, after a little algebra,

$$U^\dagger \Big(\mathcal{M} f d\Delta^\dagger + d\Delta f \mathcal{M}^\dagger \Big) U = U^\dagger \Big[(s\Delta + \mathcal{C}\Delta) f d\Delta^\dagger + d\Delta f (s\Delta + \mathcal{C}\Delta)^\dagger \Big] U\,; \tag{3.55}$$

[7] The following expression for sU would still be valid if s would represent a generic variation.

from here we conclude that

$$\mathcal{M} = s\Delta + \mathcal{C}\Delta \tag{3.56}$$

modulo "irrelevant" terms, that is terms which vanish when right (left) multiplied by U ($U^\dagger$). The same strategy as before can be repeatedly applied to the remaining equations in (2.10), thus obtaining the complete action of the BRST operator on Δ, $\mathcal{M}$, $\mathcal{C}$, $\mathcal{A}$. The result of this exercise is

$$\begin{aligned} s\Delta &= \mathcal{M} - \mathcal{C}\Delta\,, \qquad s\mathcal{M} = \mathcal{A}\Delta - \mathcal{C}\mathcal{M}\,, \\ s\mathcal{A} &= -[\mathcal{C},\mathcal{A}]\,, \qquad s\mathcal{C} = \mathcal{A} - \mathcal{C}\mathcal{C}\,, \end{aligned} \tag{3.57}$$

which is the realization of the BRST algebra on the instanton moduli space.[8]

Three observations are in order. First, it is straightforward to show that s^2 is nilpotent as it should. This can be simply done by applying once again s to each equation in (3.57). Therefore, *on instanton moduli space s is the exterior derivative*, as we announced in the previous sections. Second, the last two equations in (3.57) and the nilpotency of s suggest that $\mathcal{A}$ can be interpreted as the curvature of the connection $\mathcal{C}$ (these equations then becoming the Bianchi identity for $\mathcal{A}$ and its definition in terms of $\mathcal{C}$). Last, using the covariant derivative defined in (3.47), we can rewrite the BRST algebra on instanton moduli space in a more compact form as

$$\begin{aligned} \mathcal{S}\Delta &= \mathcal{M}\,, \qquad \mathcal{S}\mathcal{M} = \mathcal{A}\Delta\,, \\ \mathcal{S}\mathcal{A} &= 0\,, \qquad s\,\mathcal{C} + \mathcal{C}\mathcal{C} = \mathcal{A}\,. \end{aligned} \tag{3.58}$$

We now discuss the important point of how to compute the connection $\mathcal{C}$. This can be done by plugging the first equation of (3.58) into the fermionic constraint (3.19), thus getting

$$\Delta^\dagger\mathcal{C}\Delta - \left(\Delta^\dagger\mathcal{C}\Delta\right)^T = \left(\Delta^\dagger s\Delta\right)^T - \Delta^\dagger s\Delta\,. \tag{3.59}$$

It can also be shown that this equation is equivalent to the ADHM transcription of (2.26). In the following considerations we restrict our attention to the case in which $\mathcal{C}_{00} = 0$ (recall (3.48)); as we said in the last section, this is true if and only if $\mathcal{A}_{00} = 0$. The case $(\mathcal{C}_{00}, \mathcal{A}_{00}) \neq (0,0)$ is crucially different and deeply related to the fact that when the scalar field acquires a non-zero v.e.v., the theory has a new invariance (the U(1) central charge symmetry). For these reasons it will be separately analysed in section 3.3. In this section we limit ourselves to $\mathcal{C}$'s of the form

$$\mathcal{C}_f = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}'_f & \\ 0 & & & \end{pmatrix}, \tag{3.60}$$

[8] Using (3.29) and similar expressions for $\mathcal{A}\mathcal{M}$, $\mathcal{C}\Delta$, $\mathcal{C}\mathcal{M}$, the x-dependence completely disappears from (3.57).

where $\mathcal{C}'_f = -(\mathcal{C}'_f)^T$. Using the expression for the operator L introduced in (3.31), we can rewrite (3.59) more compactly as

$$L \cdot \mathcal{C}'_f = -\Lambda_{\mathcal{C}}\,, \tag{3.61}$$

where we have defined

$$\Lambda_{\mathcal{C}} \equiv \Delta^\dagger s\Delta - \left(\Delta^\dagger s\Delta\right)^T. \tag{3.62}$$

The solution to (3.61) is then formally written as

$$\mathcal{C}'_f = -L^{-1} \cdot \Lambda_{\mathcal{C}}\,, \tag{3.63}$$

due to the invertibility of L.

The rôle of the connection $\mathcal{C}$ can be conveniently elucidated by setting it to zero in (3.49); in this case the BRST algebra would read[9]

$$s\Delta = \mathcal{M}\,, \qquad s\mathcal{M} = \mathcal{A}\Delta\,, \qquad s\mathcal{A} = 0\,. \tag{3.64}$$

It can be immediately shown that in this case the operator s would fail to be nilpotent. Indeed, the action of s^2 on the ADHM matrices would become

$$s^2\Delta = \mathcal{A}\Delta\,, \qquad s^2\mathcal{M} = \mathcal{A}\mathcal{M}\,, \qquad s^2\mathcal{A} = 0\,. \tag{3.65}$$

s^2 would then be nilpotent only up to transformations generated by $k \times k$ *moduli-dependent* antisymmetric matrices, i.e. local reparametrizations in the moduli space. (3.65) are the transcription of (2.6) on the moduli space.

In summary, the universal connection $\widehat{A}$ is given by

$$\widehat{A} = U^\dagger(d + s + \mathcal{C})U\,. \tag{3.66}$$

We want now to comment on the interpretation of the results obtained in this section. The crucial observation is that, once (3.66) is given, the ADHM matrices $\mathcal{M}$ and $\mathcal{A}$ are in turn determined by (3.57) as the covariant derivative of Δ and the curvature of the connection $\mathcal{C}$, respectively; the only independent variables are the collective coordinates contained in Δ (the instanton moduli and other moduli possibly associated with redundancies of the ADHM parametrization) and their differentials (the entries of the matrix $s\Delta$). Once the reparametrization invariance has been gauged away (by giving some convenient prescription; see the explicit examples in section 4.1), physical quantities become, through their ADHM expression, differential forms[10] on the $8k$-dimensional (anti-)instanton moduli space $\mathfrak{M}^+$ ($\mathfrak{M}^-$). Operatively, this amounts to first identifying a correct parametrization for the instanton configuration (in term of the ADHM matrix Δ introduced in (3.7)), and then to computing the explicit expression for the 1-form $\mathcal{C}$ using (3.19), in which $\mathcal{M}$ is substituted by its expression (3.56). Finally, $\mathcal{A}$ is determined by the last equation in (3.57).

[9]The following transformations are in close relationship to the supersymmetry transformations of the ADHM matrices given in [4].

[10]When scalar fields have non-zero v.e.v.'s this picture is slightly modified; we postpone this discussion to section 3.3.

3.3 The BRST algebra in the ADHM formalism: the non-zero vacuum expectation value case

The realization of the BRST algebra (2.38) on instanton moduli space in the case in which scalar fields have non-vanishing v.e.v. closely parallels that of section 3.2. In particular, the universal connection $\widehat{A} = A + c + \Lambda$ is again expressed as

$$\widehat{A} = U^\dagger (d + s + \mathcal{C}) U \,, \tag{3.67}$$

and its curvature equation and Bianchi identities give rise to the same algebra (3.57) for the ADHM matrices. Notice however that in this case $\mathcal{C}$ is given by

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}' & \\ 0 & & & \end{pmatrix} , \tag{3.68}$$

the element $\mathcal{C}_{00}$ of (3.68) being related to the asymptotic behavior of the ghost $c + \Lambda$ at $|x| \to \infty$,

$$\lim_{|x|\to\infty} (c + \Lambda) \equiv \lim_{|x|\to\infty} U^\dagger (s + \mathcal{C}) U = \mathcal{C}_{00} \,. \tag{3.69}$$

From (2.43) it then follows

$$\mathcal{C}_{00} = \lambda \mathcal{A}_{00} \,. \tag{3.70}$$

The ADHM connection $\mathcal{C}$ can be calculated by solving (3.59); since

$$\Delta^\dagger \mathcal{C} \Delta - \left(\Delta^\dagger \mathcal{C} \Delta\right)^T \equiv L \cdot \mathcal{C}' + \Lambda_b(\mathcal{C}_{00}) \,, \tag{3.71}$$

then (3.59) can be written as

$$L \cdot \mathcal{C}' = -\Lambda_b(\mathcal{C}_{00}) - \Lambda_{\mathcal{C}} \,, \tag{3.72}$$

where Λ_b has been defined in (3.32), and $\Lambda_{\mathcal{C}}$ is given by (3.62). (3.72) and (3.59) are formally identical to (3.33) and (3.27), respectively; as in section 3.1, they suggest us to set

$$\mathcal{C} = \mathcal{C}_b + \mathcal{C}_f \,, \tag{3.73}$$

where $\mathcal{C}_b$ satisfies the associated homogeneous equation

$$\Delta^\dagger \mathcal{C}_b \Delta - \left(\Delta^\dagger \mathcal{C}_b \Delta\right)^T = 0 \,. \tag{3.74}$$

The solution to (3.74) is (unique and) completely specified only after imposing boundary conditions, as in (3.70). This reflects, in the ADHM language, the degeneracy (2.41) in the definition of tangent vector to the instanton moduli space, which is due to the existence of central charge transformations; we know in fact that the fermionic constraint (3.19), from which (3.59) directly follows, is just the ADHM transcription of the fermionic zero-mode equations (3.17). As in that case, the solution is unique once the non-trivial boundary condition (3.70) is imposed.[11] Let us now set

$$\mathcal{C}' = \mathcal{C}'_f + \mathcal{C}'_b \, ; \tag{3.75}$$

then (3.72) gives

$$L \cdot \mathcal{C}'_b = -\Lambda_b(\mathcal{C}_{00}) \, , \qquad L \cdot \mathcal{C}'_f = -\Lambda_{\mathcal{C}} \, , \tag{3.76}$$

whose solution is unique once $\mathcal{C}_{00}$ has been specified by means of the boundary condition (3.70). Note that, if $\mathcal{C}_{00}$ were zero, then also $\Lambda_b(\mathcal{C}_{00})$ would vanish; therefore, due to the invertibility of L, the equation for $\mathcal{C}'_b$ would only admit the trivial solution $\mathcal{C}'_b = 0$.

The matrices $\mathcal{M}$ and $\mathcal{A}$ are in turn determined by means of the ADHM algebra to be

$$\mathcal{M} = \mathcal{S}\Delta \, , \qquad \mathcal{A} = s\mathcal{C} + \mathcal{C}\mathcal{C} \, ; \tag{3.77}$$

in particular, for the (00) element of $\mathcal{A}$, we have

$$\mathcal{A}_{00} = s\mathcal{C}_{00} = \frac{\partial}{\partial\lambda}\Big(\lambda v \frac{\sigma_3}{2i}\Big) = v\frac{\sigma_3}{2i} \, , \qquad s\mathcal{A}_{00} = \frac{\partial}{\partial\lambda}\Big(v\frac{\sigma_3}{2i}\Big) = 0 \, , \tag{3.78}$$

from which it follows the expected asymptotic behavior (2.31) for the scalar field ϕ

$$\lim_{|x|\to\infty} \phi \equiv \lim_{|x|\to\infty} U^\dagger \mathcal{A} U = \mathcal{A}_{00} = v\frac{\sigma_3}{2i} \, . \tag{3.79}$$

Note that the nilpotent BRST operator s acts on the external parameters $\mathcal{C}_{00}, \mathcal{A}_{00}$, given respectively by (3.70) and (3.78), just as the partial derivative with respect to λ, while its restriction on the other elements of the ADHM matrices would act as the usual exterior derivative on the moduli space.

3.4 Algebraic construction of the BRST transformations

In this section we will derive the realization of the BRST algebra on the instanton moduli space in a direct way, i.e. using neither the BRST algebra in field space (2.10) nor the expressions for the field configuration (3.49) onto which the TYM action projects. The only ingredient we need will be a parametrization for the moduli space of instantons; in terms of the ADHM construction, this is equivalent to determine a matrix Δ which satisfies (3.7). As discussed in section 3.1, the ADHM space of parameters is acted upon by an $\mathrm{O}(k)$ reparametrization symmetry. The gauging of this symmetry will turn out to be what is required to make the BRST variations of the ADHM data Δ consistent with the algebraic constraints (3.7) which determine them. The BRST algebra on instanton moduli space, (3.57), will thus emerge as the most general set of deformations of the ADHM data compatible with (3.7).

[11] See also the discussion after (3.84).

To show this, let us now start by performing an infinitesimal scalar variation (that we call s for obvious reasons) of the bosonic constraint (3.7). We get

$$(s\Delta)^\dagger\Delta + \Delta^\dagger s\Delta = \left[(s\Delta)^\dagger\Delta\right]^T + \left(\Delta^\dagger s\Delta\right)^T . \tag{3.80}$$

This relation should be read as an equation for $s\Delta$, and we want to guess its solution. We write it as

$$s\Delta = \mathcal{M} - \mathcal{C}\Delta\,, \tag{3.81}$$

where $\mathcal{M}$ is *defined* as the matrix which satisfies (3.19). $\mathcal{C}$ is constrained by the structure of Δ (which satisfies (3.14) and (3.15) in the gauge defined by (3.13)) and $\mathcal{M}$ (which is is fixed by (3.21) and (3.22)); in conclusion, the most general expression of $\mathcal{C}$ consistent with (3.81) is

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}' & \\ 0 & & & \end{pmatrix}, \tag{3.82}$$

where $\mathcal{C}'$ is a real antisymmetric $k \times k$ matrix, $(\mathcal{C}')^\dagger = -\mathcal{C}'$. If we plug (3.81) into (3.80), the terms containing $\mathcal{M}$ exactly cancel out thanks to (3.19), whereas $\mathcal{C}$ is fixed by the equation

$$\Delta^\dagger(\mathcal{C}+\mathcal{C}^\dagger)\Delta = \left[\Delta^\dagger\left(\mathcal{C}+\mathcal{C}^\dagger\right)\Delta\right]^T, \tag{3.83}$$

which becomes

$$L\cdot\left(\mathcal{C}'+\mathcal{C}'^\dagger\right) = -\Lambda_b\left(\mathcal{C}_{00}+\mathcal{C}_{00}^\dagger\right), \tag{3.84}$$

where L is a generally invertible operator. As a consequence one must have $\mathcal{C}_{00} = -\mathcal{C}_{00}^\dagger$. One is thus led to an expression of $\mathcal{C}$ which coincides with the one previously suggested in (3.48).

Let us now pause for a moment and count the number of degrees of freedom in (3.81). On one hand, the ADHM matrix Δ and its variation $s\Delta$ both contain $8k+k(k-1)/2$ (unconstrained) degrees of freedom;[12] $\mathcal{M}$ contains instead $8k$ degrees of freedom, after solving (3.19). On the other hand, $\mathcal{C}$ contains $3+k(k-1)/2$ parameters, and we would be led to an apparent mismatch in counting the number of degrees of freedom in (3.81). Actually the three degrees of freedom introduced by $\mathcal{C}_{00}$ are not new; instead they are already included in the number of independent solutions to (3.19). This can be understood once we decompose $\mathcal{C}$ as in (3.73), where

$$\mathcal{C}_b = \begin{pmatrix} \mathcal{C}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}_b' & \\ 0 & & & \end{pmatrix} \tag{3.85}$$

[12] We recall the reader that we always work with a canonical choice of b.

is defined in such a way to satisfy the homogeneous equation (3.74). On one hand this equation is equivalent to

$$L \cdot \mathcal{C}_b' = -\Lambda_b(\mathcal{C}_{00})\,; \tag{3.86}$$

therefore it has non-trivial solutions when $\mathcal{C}_{00} \neq 0$. On the other hand, it is formally identical to (3.19) with $\mathcal{M}$ replaced by $\mathcal{C}_b\Delta$. In order to avoid double counting, its three independent solutions should then not be considered as "new". Finally, $\mathcal{C}_f$ is now constrained by (3.73), (3.81) and (3.19) to satisfy an equation identical to (3.59); thus, it just takes into account the genuinely new $k(k-1)/2$ parameters which are related to the $\mathrm{O}(k)$ reparametrization invariance. We then conclude that there is a complete balance in the number of degrees of freedom in (3.81).

If we perform the s-variation of the fermionic constraint (3.19), we get

$$(s\Delta)^\dagger\mathcal{M} + \Delta^\dagger s\mathcal{M} = \left[(s\Delta)^\dagger\mathcal{M}\right]^T + \left(\Delta^\dagger s\mathcal{M}\right)^T. \tag{3.87}$$

Taking into account (3.81), we find

$$\left(\mathcal{M}^\dagger + \Delta^\dagger\mathcal{C}\right)\mathcal{M} - \left[\left(\mathcal{M}^\dagger + \Delta^\dagger\mathcal{C}\right)\mathcal{M}\right]^T = \left(\Delta^\dagger s\mathcal{M}\right)^T - \Delta^\dagger s\mathcal{M}\,; \tag{3.88}$$

(3.88) should be thought of as an equation for $s\mathcal{M}$. Analogously to the previous case, its most general solution can be cast into the form

$$s\mathcal{M} = \mathcal{A}\Delta - \mathcal{C}\mathcal{M}\,, \tag{3.89}$$

where $\mathcal{A}$ has the same form as in (3.26). If we plug (3.89) into (3.88), we obtain that $\mathcal{A}$ must satisfy the following relation:

$$\Delta^\dagger\mathcal{A}\Delta - \left(\Delta^\dagger\mathcal{A}\Delta\right)^T = \left(\mathcal{M}^\dagger\mathcal{M}\right)^T - \mathcal{M}^\dagger\mathcal{M}\,. \tag{3.90}$$

Then (3.90) is identical to (3.27), which was obtained from a completely different point of view (the equations of motion for the scalar field ϕ), and its solution is given by (3.34), (3.35) and (3.36).

We want now to clarify the relation between $\mathcal{A}$ and $\mathcal{C}$ as defined in this section. To this end, let us perform one more s-variation of (3.81) and (3.89); after a little algebra we get

$$\begin{aligned} s^2\Delta &= \left(\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C}\right)\Delta\,, \\ s^2\mathcal{M} &= \left(\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C}\right)\mathcal{M} + \left(s\mathcal{A} + [\mathcal{C},\mathcal{A}]\right)\Delta\,. \end{aligned} \tag{3.91}$$

Once one requires the nilpotency of the BRST operator s, then

$$\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C} = 0\,, \tag{3.92}$$

$$s\mathcal{A} + [\mathcal{C},\mathcal{A}] = 0\,. \tag{3.93}$$

Therefore it is possible to interpret (3.92) as the definition of $\mathcal{A}$ as the field strength of $\mathcal{C}$ and (3.93) as its Bianchi identity. This completely clarifies the relation between $\mathcal{A}$ and $\mathcal{C}$.

In order to check the consistency of the super-constraints with the BRST variations, we still have to perform the s-variation of (3.90). If we do this, we get

$$\Delta^{\dagger}(s\mathcal{A}+[\mathcal{C},\mathcal{A}])\Delta-\left[\Delta^{\dagger}(s\mathcal{A}+[\mathcal{C},\mathcal{A}])\Delta\right]^{T}=0\,, \tag{3.94}$$

which is trivially satisfied thanks to (3.93).

Summarizing, we have found that consistency between the BRST variation of the bosonic ADHM matrix Δ and the constrain (3.7) it obeys, yields

$$\begin{aligned} s\Delta &= \mathcal{M}-\mathcal{C}\Delta\,, \qquad & s\mathcal{M} &= \mathcal{A}\Delta-\mathcal{C}\mathcal{M}\,, \\ s\mathcal{A} &= -[\mathcal{C},\mathcal{A}]\,, \qquad & s\mathcal{C} &= \mathcal{A}-\mathcal{C}\mathcal{C}\,, \end{aligned} \tag{3.95}$$

where $\mathcal{M}$ satisfies (3.19). As anticipated, this set of equations gives an explicit realization of the BRST algebra on instanton moduli space, and it coincides with that found in (3.57) with completely different methods.

4. The set-up of the calculation of instanton Green's functions

In this section we explain how to perform instanton calculations in our picture. As an application of our techniques, we will then focus on computing correlators in the case in which the relevant instanton configurations have winding number $k=1,2$. In $N=2$ SYM with non-vanishing v.e.v. for the scalar field, we will be interested in evaluating the correlator $\langle \mathrm{Tr}\,\phi^2\rangle$. These computations will show the main features of the formalism developed in the previous sections.

To make these characteristics more evident, let us now summarize the "standard" strategy to perform instanton calculations in SUSY theories [15, 21, 4, 5].

1. The action is expanded around the saddle point up to quadratic fluctuations.

2. The fields are expanded in eigenmodes and the functional measure is replaced by an integration over the coefficients of the mode expansion. The contribution of the zero-modes and that of the non-zero modes are now clearly identified.

3. The fields in the correlator are also expanded in modes and the part containing the non-zero modes is discarded since it represents higher order quantum corrections.

4. The non-zero modes are then integrated out. This integration gives a ratio of determinants which is one thanks to SUSY [27].

5. The last step consists in performing the integration over the zero-modes. In order to deal with the zero-mode sector, one has to trade integrations over the bosonic zero-modes for integrations over collective coordinates; this gives rise to a bosonic jacobian. Moreover, one has to keep into account chiral selection rules which single out the non-vanishing Green's functions. Operatively, these selection rules amount to say that all the Grassmann integrations over the fermionic collective coordinates have to be soaked up by explicitly inserting the appropriate number (say n) of zero-modes; thus, the only non-zero amplitudes will be those which admit an expansion in terms of fermion zero-modes such that the coefficient multiplying the term with n fermionic collective coordinates does not vanish. This gives rise to a fermionic jacobian, which is the determinant of the matrix whose entries are the overlaps of the fermionic zero-mode wave functions.

Our starting point will be the last step which, in the formalism of the previous sections, amounts to integrating the Lagrange multipliers in the gauge fixed TYM action (2.45). This integration naturally projects the fields A, ψ, ϕ onto the zero-modes subspace, which is identified by (3.6), (3.17) and (2.24) (supplemented by appropriate boundary conditions on ϕ);[13] the configurations which solve these equations were written in (3.49). Through these expressions, physical amplitudes will depend on Δ, $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{A}$. The ADHM equations (3.7) fix the number of independent (bosonic) collective coordinates to be $8k + k(k-1)/2$; gauge-fixing the left-over $\mathrm{O}(k)$ symmetry further reduce this number to $8k$. Moreover the first relation in (3.57) together with (3.19) allows one to compute the connection $\mathcal{C}'$ as a 1-form expanded on a basis of differentials of the bosonic moduli. If one substitutes back the computed expression for $\mathcal{C}$ into the first equation in (3.57), then the $\mathcal{M}$'s become in turn differential 1-forms on instanton moduli space $\mathfrak{M}^+$.[14] Finally (3.27) gives $\mathcal{A}$ as a function of Δ and $\mathcal{M}$. We then conclude that *any polynomial in the fields becomes, after projection onto the zero-mode subspace, a well-defined differential form on* $\mathfrak{M}^+$ [9]. We can then symbolically write

$$\langle \text{fields} \rangle = \int_{\mathfrak{M}^+} \left[(\text{fields})\, e^{-S_{\text{TYM}}} \right]_{\text{zero-mode subspace}} . \tag{4.1}$$

Let us now call $\{\widehat{\Delta}_i\}$ ($\{\widehat{\mathcal{M}}_i\}$), $i = 1, \dots, p$, where $p = 8k$, a basis of (ADHM) coordinates on $\mathfrak{M}^+$ ($T_A\mathfrak{M}^+$). (3.81) thus yields $\widehat{\mathcal{M}}_i = s\widehat{\Delta}_i + (\widehat{\mathcal{C}\Delta})_i$. A generic function

[13] If one wanted to work with anti-instantons, then (3.6) and (3.17) should be replaced by (2.22) and (2.23), respectively.

[14] Recall that the the number of independent $\mathcal{M}$'s is $8k$ (as the number of bosonic moduli) by virtue of (3.19).

on the zero-mode subspace will then have the expansion

$$
\begin{aligned}
g(\widehat{\Delta},\widehat{\mathcal{M}}) &= g_0(\widehat{\Delta}) + g_{i_1}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1} + \frac{1}{2!}g_{i_1 i_2}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2} + \cdots + \\
&\quad + \frac{1}{p!}g_{i_1 i_2 \cdots i_p}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p}\,, \qquad (4.2)
\end{aligned}
$$

the coefficients of the expansion being totally antisymmetric in their indices. Now the first of (3.57) implies that the $\widehat{\mathcal{M}}_i$'s and the $s\widehat{\Delta}_i$'s are related by a (moduli-dependent) linear transformation K_{ij}, which is completely known once the explicit expression for $\mathcal{C}$ is plugged into the $\widehat{\mathcal{M}}_i$'s:

$$
\widehat{\mathcal{M}}_i = K_{ij}(\widehat{\Delta})s\widehat{\Delta}_j\,. \qquad (4.3)
$$

It then follows that

$$
\begin{aligned}
\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p} &= K_{i_1 j_1}K_{i_2 j_2}\cdots K_{i_p j_p}s\widehat{\Delta}_{j_1}s\widehat{\Delta}_{j_2}\cdots s\widehat{\Delta}_{j_p} \\
&= \epsilon_{j_1\cdots j_p}K_{i_1 j_1}K_{i_2 j_2}\cdots K_{i_p j_p}s^p\widehat{\Delta} \\
&= \epsilon_{i_1\cdots i_p}(\mathrm{det}K)s^p\widehat{\Delta}\,, \qquad (4.4)
\end{aligned}
$$

where $s^p\widehat{\Delta} \equiv s\widehat{\Delta}_1\cdots s\widehat{\Delta}_p$. From (4.2) and (4.3) we conclude that

$$
\begin{aligned}
\int_{\mathcal{M}^+} g(\widehat{\Delta},\widehat{\mathcal{M}}) &= \frac{1}{p!}\int_{\mathcal{M}^+} g_{i_1 i_2\cdots i_p}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p} \\
&= \int_{\mathcal{M}^+} s^p\widehat{\Delta}|\mathrm{det}K|g_{12\cdots p}(\widehat{\Delta})\,. \qquad (4.5)
\end{aligned}
$$

This formula is an operative tool to calculate physical amplitudes. Here the determinant of K naturally stands out as *the instanton integration measure for* $N=2$ *SYM theories.* This important ingredient of the calculation is obtained in standard instanton calculations as a ratio of bosonic and fermionic zero-mode jacobians. Instead, in our approach it emerges in a geometrical and very direct way, *without the need of any computation of ratios of determinants, nor of any knowledge of the explicit expressions of bosonic and fermionic zero-modes.* The only ingredient is the connection $\mathcal{C}$. As an instructive exercise, in the following subsection we will compute K and its determinant (i.e. the instanton measure) in the cases of winding number equal to one and two. We anticipate that the results we get will agree with previously known formulae; however they are obtained here in a very quick and straightforward way.

A last remark concerns what happens to the action of the theory, S_{TYM}, when it is restricted to the zero-mode subspace (we called the corresponding expression S_{inst} for obvious reasons). In section 2.2 we saw that for $v \neq 0$ it is non vanishing; its expression was given in (2.50). In the following we will need to explicitly compute S_{inst} as a function of the instanton moduli; this will be done in section 4.2, where we will also be able to write it as a total BRST derivative.

4.1 The instanton measure for winding number $k = 1, 2$

The ADHM bosonic and fermionic matrices can be written as

$$\Delta = \begin{pmatrix} w \\ x_0 - x \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} \mu \\ \xi \end{pmatrix}. \tag{4.6}$$

For $k = 1$ there are no constraints over the collective coordinates; therefore the left-over reparametrization group introduced in (3.12) is trivial. As a result we simply have

$$\mathcal{M} = \begin{pmatrix} sw \\ sx_0 \end{pmatrix}, \tag{4.7}$$

and $\det K = 1$. The instanton measure is then given by $s^4 x_0 s^4 w$, which is the well-known 't Hooft measure [25]. We now move on to the more interesting case of $k = 2$.

The ADHM bosonic matrix reads

$$\Delta = \begin{pmatrix} w_1 & w_2 \\ x_1 - x & a_1 \\ a_1 & x_2 - x \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix} + b(x - x_0)\,, \tag{4.8}$$

where $x_0 = (x_1 + x_2)/2$, $a_3 = (x_1 - x_2)/2$. We also need the expression of the matrix $\mathcal{M}$ which is defined in (3.19). Since this constraint is very similar to (3.7) (to get convinced of this fact just think that two solutions of (3.19) are given by $\mathcal{M}$ proportional to a and b) it is convenient to choose a form of $\mathcal{M}$ which parallels (4.8), i.e.

$$\mathcal{M} = \begin{pmatrix} \mu_1 & \mu_2 \\ \xi + \mathcal{M}_3 & \mathcal{M}_1 \\ \mathcal{M}_1 & \xi - \mathcal{M}_3 \end{pmatrix} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mathcal{M}_3 & \mathcal{M}_1 \\ \mathcal{M}_1 & -\mathcal{M}_3 \end{pmatrix} - b\xi\,. \tag{4.9}$$

The solution to the bosonic constraint (3.7) is simply given by

$$a_1 = \frac{1}{4|a_3|^2} a_3 \Big(\bar{w}_2 w_1 - \bar{w}_1 w_2 + \Sigma \Big)\,, \tag{4.10}$$

where Σ is an arbitrary real parameter related to the left-over O(2) symmetry. In the following we will exploit this O(2) gauge freedom to put Σ to zero. The constraint (3.19) is satisfied imposing

$$\mathcal{M}_1 = \frac{a_3}{2|a_3|^2} \Big(2\bar{a}_1 \mathcal{M}_3 + \bar{w}_2 \mu_1 - \bar{w}_1 \mu_2 \Big)\,; \tag{4.11}$$

from now on we will choose $\{\mu_1, \mu_2, \xi, \mathcal{M}_3\}$ as a set of independent fermionic variables. Finally, the equation $L \cdot \mathcal{A}' = -\Lambda_b - \Lambda_f$ reduces to

$$H(\mathcal{A}'_f)_{12} = (\Lambda_f)_{12} \equiv \bar{\mu}_1 \mu_2 - \bar{\mu}_2 \mu_1 + 2\Big(\bar{\mathcal{M}}_3 \mathcal{M}_1 - \bar{\mathcal{M}}_1 \mathcal{M}_3 \Big)\,, \tag{4.12}$$

$$H(\mathcal{A}'_b)_{12} = (\Lambda_b)_{12} \equiv \bar{w}_1 \mathcal{A}_{00} w_2 - \bar{w}_2 \mathcal{A}_{00} w_1\,, \tag{4.13}$$

where $H = |w_1|^2+|w_2|^2+4(|a_3|^2+|a_1|^2)$. Let us now write the BRST transformations of the bosonic ADHM variables:

$$\begin{aligned}
\mu_1 &= sw_1 + \mathcal{C}_{12}w_2 + \mathcal{C}_{00}w_1\,, \qquad \mu_2 = sw_2 - \mathcal{C}_{12}w_1 + \mathcal{C}_{00}w_2\,,\\
\xi &= sx_0\,, \qquad \mathcal{M}_3 = sa_3 + 2\mathcal{C}_{12}a_1\,, \qquad \mathcal{M}_1 = sa_1 - 2\mathcal{C}_{12}a_3\,.
\end{aligned} \tag{4.14}$$

The component $\mathcal{C}_{12}$ of the O(2) connection

$$\mathcal{C}' = \begin{pmatrix} 0 & \mathcal{C}_{12} \\ -\mathcal{C}_{12} & 0 \end{pmatrix} \tag{4.15}$$

can be simply obtained by plugging the right-hand sides of (4.14) into the fermionic constraint $(\Delta^\dagger\mathcal{M})_{12} = (\Delta^\dagger\mathcal{M})_{21}$ and solving for $\mathcal{C}_{12}$. Actually, the terms containing $\mathcal{C}_{00}$, which is given by (3.70), can be discarded, since they do not contribute upon integration on the instanton moduli space. This way we get

$$\mathcal{C}_{12} = \frac{1}{H}\Big[\bar{w}_1 sw_2 - \bar{w}_2 sw_1 + 2(\bar{a}_3 sa_1 - \bar{a}_1 sa_3)\Big]\,. \tag{4.16}$$

Eliminating sa_1 via (4.10) (in the gauge $\Sigma = 0$), one can rewrite $\mathcal{C}_{12}$ in terms of differentials of independent bosonic moduli, thus obtaining

$$\mathcal{C}_{12} = \frac{1}{2H}\Big[\bar{w}_1 sw_2 - \bar{w}_2 sw_1 - 4\bar{a}_1 sa_3 + s\bar{w}_2 w_1 - s\bar{w}_1 w_2 - 4s\bar{a}_3 a_1\Big]\,. \tag{4.17}$$

Two observations are in order. First, we remark that (4.17) clearly shows that $\mathcal{C}_{12}$ is real, as a connection of an orthogonal group should. Moreover, the r.h.s. of (4.17) does not depend on sx_0; for this reason and from (4.14) it immediately follows that in computing $\det K$ we can discard the variable $\xi \equiv sx_0$, which would contribute with the determinant of a unit matrix. We will then define a "reduced" fermionic matrix (of quaternions) $\widetilde{\mathcal{M}}$ as

$$\widetilde{\mathcal{M}} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mathcal{M}_3 \end{pmatrix}, \tag{4.18}$$

its bosonic counterpart being

$$\widetilde{\Delta} = \begin{pmatrix} w_1 \\ w_2 \\ a_3 \end{pmatrix}. \tag{4.19}$$

The relation between $\widetilde{\mathcal{M}}$ and $s\widetilde{\Delta}$ can be cast into the form

$$\left(\widetilde{\mathcal{M}}_{\alpha\dot\alpha}\right)_i = \sigma^\mu_{\alpha\dot\alpha}(K_{\mu\nu})_{ij}\left(s\widetilde{\Delta}_\nu\right)_j\,, \tag{4.20}$$

where $i = 1,2,3$. Plugging (4.17) into (4.14), we get, after a little algebra, the following explicit expression for K,

$$(K_{\mu\nu})_{ij} = \begin{pmatrix} \delta_{\mu\nu} - w_{2\mu}w_{2\nu}/H & w_{2\mu}w_{1\nu}/H & -4w_{2\mu}a_{1\nu}/H \\ w_{1\mu}w_{2\nu}/H & \delta_{\mu\nu} - w_{1\mu}w_{1\nu}/H & 4w_{1\mu}a_{1\nu}/H \\ -2a_{1\mu}w_{2\nu}/H & 2a_{1\mu}w_{1\nu}/H & \delta_{\mu\nu} - 8a_{1\mu}a_{1\nu}/H \end{pmatrix}, \tag{4.21}$$

whose determinant we want now to compute.

To this end, let us write K as

$$K = \mathbb{1} - zz^T P = \left(P^{-1} - zz^T\right) P\,, \tag{4.22}$$

where

$$P = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & 2\cdot\mathbb{1} \end{pmatrix} \tag{4.23}$$

and

$$z = \frac{1}{\sqrt{H}} \begin{pmatrix} w_2 \\ -w_1 \\ 2a_1 \end{pmatrix}. \tag{4.24}$$

It is easy to verify that the determinant of a matrix of the form

$$Q = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) - zz^T\,, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \tag{4.25}$$

is simply

$$\det Q = \prod_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \left(\prod_{j\neq i} \alpha_j\right) |z_i|^2\,, \tag{4.26}$$

from which it is straightforward to get

$$|\det K| = \frac{4\Big||a_3|^2 - |a_1|^2\Big|}{H}. \tag{4.27}$$

Restoring the Σ dependence of a_1 and sa_1 in (4.17) we obtain

$$\mathcal{C}_{12} = \frac{1}{2H}\Big[\bar{w}_1 s w_2 - \bar{w}_2 s w_1 - 4\bar{a}_1 s a_3 + s\bar{w}_2 w_1 - s\bar{w}_1 w_2 - 4s\bar{a}_3 a_1 + s\Sigma\Big]\,, \tag{4.28}$$

where $\Sigma = \Sigma(w_1, w_2, a_3, x_0)$. $\mathcal{C}_{12}$ contains also a term proportional to sx_0; however, this term turns out not to contribute to $\det K$ and, in fact, we find

$$|\det K| = \frac{4}{H}\left| |a_3|^2 - |a_1|^2 + \frac{1}{4}\frac{\partial\Sigma}{\partial a_{3\mu}} a_{1\mu} + \frac{1}{8}\frac{\partial\Sigma}{\partial w_{1\mu}} w_{2\mu} - \frac{1}{8}\frac{\partial\Sigma}{\partial w_{2\mu}} w_{1\mu}\right|. \tag{4.29}$$

It is now possible to write the terms containing Σ in a more compact way. To this end, recall that the action of the $\mathrm{O}(2)$ reparametrization group on (w_1, w_2, a_3, a_1) can be read from (3.12) and (3.16) with $q = \mathbb{1}$ and

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.30}$$

It is straightforward to show that

$$a_{1\mu} = -\frac{1}{2}\frac{\partial a^\theta_{3\mu}}{\partial\theta}\bigg|_{\theta=0}\,, \qquad w_{2\mu} = -\frac{\partial w^\theta_{1\mu}}{\partial\theta}\bigg|_{\theta=0}\,, \qquad w_{1\mu} = \frac{\partial w^\theta_{2\mu}}{\partial\theta}\bigg|_{\theta=0}\,, \tag{4.31}$$

so that we can finally rewrite (4.29) as

$$|\det K| = \frac{4\left|\,|a_3|^2 - |a_1|^2 - \frac{1}{8}\frac{\partial\Sigma^\theta}{\partial\theta}\Big|_{\theta=0}\right|}{H}\,. \tag{4.32}$$

This result exactly gives the instanton measure for $N = 2$ pure SYM, which was obtained in [4] as a ratio of fermionic and bosonic jacobians. In our formalism it is simply the determinant of a coordinate transformation, and it is possible to write it down with the precise knowledge of $\mathcal{C}$ alone. We will further clarify the rôle of $\mathcal{C}$ in the last section of this work, where we will construct the moduli space of self-dual gauge connections as a hyperkähler quotient space.

4.2 The multi-instanton action from the TYM action

When restricted to the zero-mode subspace, the TYM action vanishes up to the boundary term written in (2.50), which leads to the non-trivial multi-instanton action

$$S_{\text{inst}} = s\int d^4x\, 2\partial^\mu\, \mathrm{Tr}\left(\bar\phi\psi_\mu\right) = 4\pi^2 \lim_{|x|\to\infty} |x|^3 \frac{x^\mu}{|x|} s\, \mathrm{Tr}\left(\bar\phi\psi_\mu\right). \tag{4.33}$$

In the limit $|x| \to \infty$, the only non-vanishing term is given by $\mathrm{Tr}[\bar\phi(s\psi_\mu)]$. Let us now calculate the asymptotic limits of $\bar\phi$ and ψ_μ for winding number k. For the scalar field we trivially have $\lim_{|x|\to\infty}\bar\phi = \bar{\mathcal{A}}_{00}$, while for ψ_μ

$$\lim_{|x|\to\infty}\psi_\mu = \lim_{|x|\to\infty}\left(U^\dagger\mathcal{M}fb\bar\sigma_\mu U - U^\dagger b\sigma_\mu f\mathcal{M}^\dagger U\right). \tag{4.34}$$

Knowing that asymptotically

$$U_0 \longrightarrow \sigma_0\,, \qquad U_k \longrightarrow -\frac{1}{|x|^2}x\bar w_k U_0\,, \qquad f_{kl} \longrightarrow \frac{1}{|x|^2}\delta_{kl}\,, \tag{4.35}$$

we get

$$\begin{aligned}
\lim_{|x|\to\infty} x^\mu\psi_\mu &= \lim_{|x|\to\infty} x^\mu \sum_{p,l,m=1}^{k}\left[U_0^\dagger\mu_p\left(\frac{1}{|x|^2}\delta_{pl}\right)b_{lm}\bar\sigma_\mu U_m - U_p^\dagger b_{pl}\sigma_\mu\left(\frac{1}{|x|^2}\delta_{lm}\right)\bar\mu_m U_0\right]\\
&= \frac{1}{|x|^2}\sum_{l=1}^{k}(\mu_l\bar w_l - w_l\bar\mu_l)\,;
\end{aligned} \tag{4.36}$$

from this we conclude that

$$\begin{aligned}
[S_{\text{inst}}]_k &= 4\pi^2 s\,\mathrm{Tr}\left[\bar{\mathcal{A}}_{00}\sum_{l=1}^{k}(\mu_l\bar w_l - w_l\bar\mu_l)\right] \qquad (4.37)\\
&= 4\pi^2 s\,\mathrm{Tr}\left[\bar{\mathcal{A}}_{00}\left(\mathcal{M}\Delta^\dagger - \Delta\mathcal{M}^\dagger\right)_{00}\right] = -4\pi^2 s\,\mathrm{Tr}\left[\bar{\mathcal{A}}_{00}\left(\Delta\overleftrightarrow{\mathcal{S}}\Delta^\dagger\right)_{00}\right],
\end{aligned}$$

which explicitly gives the instanton action as the total BRST variation of a function of the bosonic and fermionic collective coordinates. In the second equality, the subscript 00 stands for the upper left entry of the matrix in parentheses, and $\mathcal{S}$ is the covariant derivative on instanton moduli space defined in (3.47). Note that only some moduli are involved in this expression, more precisely, the unconstrained ones.

It is easy to convince oneself that (4.37) reproduces the instanton action for $N = 2$ SYM as written in [4]. To this end, let us now act with the operator s on the moduli. From (3.57) and (3.29) it follows that

$$\begin{aligned} s w_l &= \mu_l - \left(\mathcal{C}_{00} w_l - \sum_{p=1}^{k} w_p \mathcal{C}'_{pl} \right), \\ s \mu_l &= \mathcal{A}_{00} w_l - \sum_{p=1}^{k} w_p \mathcal{A}'_{pl} - \mathcal{C}_{00} \mu_l ; \end{aligned} \tag{4.38}$$

we get

$$\begin{aligned} s \operatorname{Tr}\left[\bar{\mathcal{A}}_{00} \sum_{l=1}^{k} (\mu_l \bar{w}_l - w_l \bar{\mu}_l) \right] = \operatorname{Tr}\Bigg[& 2(\bar{\mathcal{A}}_{00}\mathcal{A}_{00}) \sum_{l=1}^{k} |w_l|^2 - 2\bar{\mathcal{A}}_{00} \sum_{l=1}^{k} \mu_l \bar{\mu}_l - \\ & - 2\bar{\mathcal{A}}_{00} \sum_{l,p=1}^{k} w_l \mathcal{A}'_{lp} \bar{w}_p + \\ & + \sum_{l=1}^{k} \left(-\bar{\mathcal{A}}_{00}\mathcal{C}_{00} + \mathcal{C}_{00}\bar{\mathcal{A}}_{00} \right) \mu_l \bar{w}_l + \\ & + \sum_{l=1}^{k} \left(\bar{\mathcal{A}}_{00}\mathcal{C}_{00} - \mathcal{C}_{00}\bar{\mathcal{A}}_{00} \right) w_l \bar{\mu}_l \Bigg], \end{aligned} \tag{4.39}$$

where we also used the fact that $s\mathcal{A}_{00} = 0$. The last two terms in (4.39) vanish by virtue of (3.70); therefore, we conclude that

$$\begin{aligned} [S_{\rm inst}]_k = 4\pi^2 \operatorname{Tr}\Bigg[& 2(\bar{\mathcal{A}}_{00}\mathcal{A}_{00}) \sum_{l=1}^{k} |w_l|^2 - 2\bar{\mathcal{A}}_{00} \sum_{l=1}^{k} \mu_l \bar{\mu}_l + \\ & + \sum_{l,p=1}^{k} (\bar{w}_l \bar{\mathcal{A}}_{00} w_p - \bar{w}_p \bar{\mathcal{A}}_{00} w_l) \mathcal{A}'_{lp} \Bigg] \end{aligned} \tag{4.40}$$

which exactly reproduces the $N = 2$ SYM action in moduli space[15] [4]. $[S_{\rm inst}]_k$ can be decomposed as $[S_{\rm inst}]_k = [S_B]_k + [S_F]_k$, where $[S_B]_k$ ($[S_F]_k$) is the Higgs action (the Yukawa action) for instanton number k. Explicitly

[15]Note that in our notations $2\operatorname{Tr}(\bar{\mathcal{A}}_{00}\mathcal{A}_{00}) = |v|^2$.

$$[S_B]_k = 4\pi^2 \,\mathrm{Tr}\left[2(\bar{\mathcal{A}}_{00}\mathcal{A}_{00})\sum_{l=1}^{k}|w_l|^2 + \sum_{l,p=1}^{k}(\bar{w}_l\bar{\mathcal{A}}_{00}w_p - \bar{w}_p\bar{\mathcal{A}}_{00}w_l)(\mathcal{A}'_b)_{lp}\right], \tag{4.41}$$

$$[S_F]_k = 4\pi^2 \,\mathrm{Tr}\left[-2\bar{\mathcal{A}}_{00}\sum_{l=1}^{k}\mu_l\bar{\mu}_l + \sum_{l,p=1}^{k}(\bar{w}_l\bar{\mathcal{A}}_{00}w_p - \bar{w}_p\bar{\mathcal{A}}_{00}w_l)(\mathcal{A}'_f)_{lp}\right], \tag{4.42}$$

$\mathcal{A}'_b$ and $\mathcal{A}'_f$ being defined in (3.34).

We are now ready to perform explicit instanton calculations in our framework.

5. Computation of instanton-dominated correlators in the Seiberg-Witten model

The strategy for computing instanton-dominated correlators in our set-up has been described at the beginning of section 4. Here we focus the attention on the Green's function $\langle \mathrm{Tr}\,\phi^2\rangle$, which is relevant for the computation of the Seiberg-Witten low-energy effective action [17, 5]. To begin with, notice that the group of translations in $\mathbb{R}^4$ is a symmetry of the theory even in the case $v \neq 0$; as a consequence x_0 and its supersymmetric counterpart $sx_0 \equiv \xi$ (which is naturally expressed as the BRST variation of the instanton configuration center x_0) do not appear in $S_{\rm inst}$, as a direct check of (4.40) also shows. They will then have to be soaked up by selecting the translational part in the correlator insertion $\mathrm{Tr}\,\phi^2$; this amounts to performing the replacement $\phi \to F_{\mu\nu}\cdot(1/2)(sx_{0\mu}sx_{0\nu})$, i.e.

$$\mathrm{Tr}\,\phi^2 \longrightarrow \frac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})s^4x_0\,. \tag{5.1}$$

The integral over these collective coordinates can now be easily performed giving the winding number [5],

$$\int_{\{x_0\}}\frac{\mathrm{Tr}\,\phi^2}{8\pi^2} \longrightarrow -k\,. \tag{5.2}$$

Therefore, we get

$$\left\langle\frac{\mathrm{Tr}\,\phi^2}{8\pi^2}\right\rangle_k = -k\int_{\mathcal{M}^+\backslash\{x_0\}} e^{-[S_{\rm inst}]_k}\,. \tag{5.3}$$

The last step consists in integrating the exponential of the instanton action over the remaining collective coordinates, i.e. over the "reduced" moduli space $\mathcal{M}^+\backslash\{x_0\}$, whose dimension is $4n$ where $n = 2k-1$. Let us call $\widetilde{\Delta}_i$, $i = 1,\dots,n$ the ADHM data for $\mathcal{M}^+\backslash\{x_0\}$, and $\widetilde{\mathcal{M}}_i$, $i = 1,\dots,n$ their fermionic counterpart (therefore ξ is not included in the $\widetilde{\mathcal{M}}_i$'s). The $\widetilde{\Delta}_i$'s and the $\widetilde{\mathcal{M}}_i$'s are respectively the generalizations of (4.19) and (4.18) for instanton number k. After substituting the solutions to the fermionic constraint (3.19) in (4.42), $[S_F]_k$ can be written as

$$[S_F]_k = \overline{\widetilde{\mathcal{M}}}_i^{\dot{A}\alpha}(h_{ij})_\alpha{}^\beta\left(\widetilde{\mathcal{M}}_j\right)_{\beta\dot{A}}, \tag{5.4}$$

where $i,j = 1,\ldots,n$ and[16] $h = -h^\dagger$; let us also define $h_{ij} = 8\pi^2 \widehat{h}_{ij}$, for the sake of future convenience. In the $k=1$ case one simply has $\widehat{h} = \bar{v}$, whereas for $k=2$ the explicit expression for $\widehat{h}_{ij}$ will be written in (5.15).

The exponential of $[S_F]_k$ can now be expanded in powers. Under the integration over the reduced moduli space $\mathcal{M}^+\backslash\{x_0\}$ the only surviving term of the expansion will be the one that, after using (3.56), produces the top form on $\mathcal{M}^+\backslash\{x_0\}$. It is crucial to remark that all the terms containing $\mathcal{C}_{00}$ do not contribute to the amplitudes since the parameter λ introduced in (2.34) does not belong to the moduli space. In order to better perform this expansion, let us define

$$\widetilde{\mathcal{M}}_i = \begin{pmatrix} (\widetilde{\mathcal{M}}_i)_4 + i(\widetilde{\mathcal{M}}_i)_3 & i(\widetilde{\mathcal{M}}_i)_1 + (\widetilde{\mathcal{M}}_i)_2 \\ i(\widetilde{\mathcal{M}}_i)_1 - (\widetilde{\mathcal{M}}_i)_2 & (\widetilde{\mathcal{M}}_i)_4 - i(\widetilde{\mathcal{M}}_i)_3 \end{pmatrix} = \begin{pmatrix} (\eta_i)_1 & -(\bar{\eta}_i)_2 \\ (\eta_i)_2 & (\bar{\eta}_i)_1 \end{pmatrix}, \tag{5.5}$$

where $(\widetilde{\mathcal{M}}_i)_\mu$, $\mu = 1,\ldots,4$ are the cartesian components of $\widetilde{\mathcal{M}}_i$. (5.4) can then be written as

$$\begin{aligned} [S_F]_k &= \bar{\eta}_i^\alpha \Big[(h_{ij})_\alpha{}^\beta + \epsilon_{\alpha\delta} (h_{ji})_\sigma{}^\delta \epsilon^{\beta\sigma} \Big] (\eta_j)_\beta = \bar{\eta}_i^\alpha \left[\left(h - h^\dagger \right)_{ij} \right]_\alpha{}^\beta (\eta_j)_\beta \\ &= 2\bar{\eta}_i^\alpha (h_{ij})_\alpha{}^\beta (\eta_j)_\beta \,, \end{aligned} \tag{5.6}$$

since h is anti-hermitean. In order to recognize the coefficient of the top form, we now explicitly expand $\exp\left(-[S_F]_k\right)$. After a little algebra, one finds

$$e^{-[S_F]_k}\Big|_{\text{top form}} = (32\pi^2)^{2n} \det \widehat{h} \prod_{i=1}^n \left[\left(\widetilde{\mathcal{M}}_i\right)_1 \left(\widetilde{\mathcal{M}}_i\right)_2 \left(\widetilde{\mathcal{M}}_i\right)_3 \left(\widetilde{\mathcal{M}}_i\right)_4 \right], \tag{5.7}$$

where we used (5.6) and $\eta_1\eta_2\bar{\eta}_1\bar{\eta}_2 = -4\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_2\widetilde{\mathcal{M}}_3\widetilde{\mathcal{M}}_4$. The coefficient of the top form on the reduced moduli space is then proportional to the determinant of the matrix H. However, one more ingredient now emerges: the matrix K of the change of coordinate basis between $\mathcal{M}_i$ and $s\Delta_i$; recalling (4.4), we conclude in fact that

$$e^{-[S_F]_k}\Big|_{\text{top form}} = (32\pi^2)^{2n} \det \widehat{h} |\det K| s^{4n} \widetilde{\Delta} \,. \tag{5.8}$$

Finally, inserting (5.8) in (5.3) we get

$$\left\langle \frac{\operatorname{Tr}\phi^2}{8\pi^2} \right\rangle_k = -k \cdot (32\pi^2)^{2n} \int_{\mathcal{M}^+\backslash\{x_0\}} s^{4n} \widetilde{\Delta} \det \widehat{h} |\det K| e^{-[S_B]_k} \,, \tag{5.9}$$

where $[S_B]_k$ is written in (4.41). Note that in (5.8) and (5.9) the instanton integration measure $|\det K|$ has naturally come out.

This is our starting point. Let us now perform the $k=1$ computation explicitly.

[16]In the following equation we denote by $h^\dagger$ the hermitean conjugate matrix obtained *without* complex conjugating v, i.e. treating v as real.

5.1 The $k = 1$ case in the bulk

In the $k = 1$ case, the structure of the instanton moduli space $\mathcal{M}^+$ has been thoroughly investigated, and it is explicitly known to be the manifold $\mathbb{R}^4 \times \mathbb{R}^+ \times S^3/\mathbb{Z}_2$ [34, 41]; the three factors correspond respectively to the instanton center (x_0), scale ($|w|$) and orientation in color space ($w/|w|$). The reduced moduli space $\mathcal{M}^+ \backslash \{x_0\}$ is then the 4-dimensional manifold $\mathbb{R}^+ \times S^3/\mathbb{Z}_2$, obtained after first integrating out the collective coordinate x_0 [5].

The ADHM bosonic and fermionic matrices are written in (4.6), and the action (4.40) calculated on the one-instanton background is given by

$$[S_{\text{inst}}]_{k=1} = 4\pi^2 \Big[|v|^2 |w|^2 - 2\,\mathrm{Tr}(\bar{v}\mu\bar{\mu}) \Big] . \tag{5.10}$$

Taking into account that

$$\mathrm{Tr}(\mu\bar{\mu}\bar{v}) = -s w_\mu s w_\nu \sum_{a=1}^{3} \eta^a_{\mu\nu} (v^a)^* , \tag{5.11}$$

where $\eta^a_{\mu\nu}$ are the 't Hooft symbols, we get, after a little algebra,

$$\begin{aligned} \left\langle \frac{\mathrm{Tr}\,\phi^2}{8\pi^2} \right\rangle_{k=1} &= -\int_{\mathcal{M}^+\backslash\{x_0\}} e^{-[S_{\text{inst}}]_{k=1}} = -\int_{\mathcal{M}^+\backslash\{x_0\}} e^{-4\pi^2\left[|v|^2|w|^2 - 2\mathrm{Tr}(\mu\bar{\mu}\bar{v})\right]} \\ &= -\frac{(8\pi^2)^2}{2!} \int_{\mathcal{M}^+\backslash\{x_0\}} e^{-4\pi^2|v|^2|w|^2}\, \mathrm{Tr}(\mu\bar{\mu}\bar{v})\, \mathrm{Tr}(\mu\bar{\mu}\bar{v}) \\ &= -(8\pi^2)^2 \cdot 2^2 \int_{\mathcal{M}^+\backslash\{x_0\}} (v^*)^2 e^{-4\pi^2|v|^2|w|^2} s^4 w = -\frac{8\pi^2}{v^2} , \end{aligned} \tag{5.12}$$

which is the expected result [5]. In (5.10) and hereafter we use the shorthand notation $\bar{v} = \bar{\mathcal{A}}_{00} = -v^* \sigma^3/2i$.

5.2 The $k = 2$ case in the bulk

In this subsection we will describe the $k = 2$ computation in the bulk i.e. without using the property that the action is BRST exact. All the features of the topological approach will now become apparent.

The action (4.40) calculated on the two-instanton background is given by

$$\begin{aligned} [S_{\text{inst}}]_{k=2} &= [S_B + S_F]_{k=2} \\ &= 4\pi^2 |v|^2 (|w_1|^2 + |w_2|^2) - 16\pi^2 \frac{|\omega|^2}{H} + \\ &\quad + 8\pi^2\, \mathrm{Tr} \left\{ \bar{\mu}_1 \bar{v} \mu_1 + \bar{\mu}_2 \bar{v} \mu_2 + \frac{\bar{\omega}}{H} \Big[\bar{\mu}_1 \mu_2 - \bar{\mu}_2 \mu_1 + 2 \left(\bar{\mathcal{M}}_3 \mathcal{M}_1 - \bar{\mathcal{M}}_1 \mathcal{M}_3 \right) \Big] \right\} , \end{aligned} \tag{5.13}$$

where we have defined

$$\omega = -\Lambda_{b12}(\mathcal{A}_{00}) = -\bar{w}_1 \mathcal{A}_{00} w_2 + \bar{w}_2 \mathcal{A}_{00} w_1 . \tag{5.14}$$

After substituting the fermionic constraint (4.11) in (5.13), $[S_F]_{k=2}$ can be written as in (5.4); the indices i, j run from 1 to 3, the $\widetilde{\mathcal{M}}_i$'s are defined in (4.18), and $h_{ij} = 8\pi^2 \widehat{h}_{ij}$, where explicitly

$$\widehat{h}_{ij} = \begin{pmatrix} \bar{v} & \frac{\bar{\omega}}{H} & -\frac{\bar{\omega}}{H|a_3|^2} w_2 \bar{a}_3 \\ -\frac{\bar{\omega}}{H} & \bar{v} & \frac{\bar{\omega}}{H|a_3|^2} w_1 \bar{a}_3 \\ \frac{\bar{\omega}}{H|a_3|^2} a_3 \bar{w}_2 & -\frac{\bar{\omega}}{H|a_3|^2} a_3 \bar{w}_1 & \frac{2\bar{\omega}}{H|a_3|^2} (a_3 \bar{a}_1 - a_1 \bar{a}_3) \end{pmatrix}. \tag{5.15}$$

Moreover, specializing (5.8) to the $k = 2$ case we get

$$e^{-[S_F]_{k=2}}\Big|_{\text{top form}} = (32\pi^2)^6 \det \widehat{h} | \det K | s^4 w_1 s^4 w_2 s^4 a_3 \,. \tag{5.16}$$

The determinant of the matrix K in (4.21) was explicitly computed in (4.27). We want now to calculate the determinant of $\widehat{h}$. To this end, note that this matrix has the form[17]

$$\widehat{h} = \begin{pmatrix} F & B & C \\ -B & F & D \\ -\bar{C} & -\bar{D} & E \end{pmatrix}, \tag{5.17}$$

where $\bar{F} = -F$ and $B = \bar{B}$. By means of elementary operations on the rows and columns of the matrix $\widehat{h}$ (i.e. by multiplying rows by quaternions and then adding and subtracting rows) we can write

$$\widehat{h} = h_1 h_2 = \begin{pmatrix} \frac{F}{|F|^2|B|^2} & \frac{F}{|B|^2}\left(\frac{1}{|F|^2} - \frac{\alpha\beta^{-1}}{|C|^2}\right) & -\frac{FC}{|C|^2} \\ 0 & \frac{B\alpha\beta^{-1}}{|B|^2|C|^2} & \frac{BC}{|C|^2} \\ 0 & 0 & 1 \end{pmatrix} \times$$
$$\times \begin{pmatrix} 0 & 0 & \gamma - \beta\alpha^{-1}\delta \\ 0 & \beta & \beta\alpha^{-1}\delta \\ -\bar{C} & -\bar{D} & E \end{pmatrix}, \tag{5.18}$$

where

$$\alpha = |C|^2 \bar{B} F + |B|^2 C \bar{D}\,, \qquad \beta = |B|^2 \bar{F} B + |F|^2 \bar{B} F\,,$$
$$\gamma = |B|^2 \bar{F} C + |F|^2 \bar{B} D\,, \qquad \delta = |C|^2 \bar{B} D - |B|^2 C E\,. \tag{5.19}$$

Using (5.18) one finds, after some algebra,

$$\det \widehat{h} = \left(\frac{\bar{\omega} v^*}{2H}\right)^2 \frac{1}{|a_3|^4} \det \left\{ \frac{v^*}{2}(\bar{w}_1 w_2 - \bar{w}_2 w_1) - \frac{i\bar{\omega}}{H}\left(\bar{w}_1 \sigma^3 w_1 + \bar{w}_2 \sigma^3 w_2\right) \right\}, \tag{5.20}$$

where σ^3 is the third Pauli matrix. (5.20) is the determinant of a quaternion, i.e. the squared absolute value of the quaternion itself. The final result is

$$\det \widehat{h} = \left(\frac{\bar{\omega} v^*}{2H}\right)^2 \frac{1}{|a_3|^4} \left\{ 2\bar{\omega}^2 \tau_1 + \left(\frac{v^*}{2}\right)^2 |\Omega|^2 + \bar{\omega}^2 \left[\tau_2 + \frac{1}{(v^*/2)^2} \left(\frac{\bar{\omega}}{H}\right)^2 \right] \right\}, \tag{5.21}$$

[17] In the following equation we denote by a bar over a quaternion the hermitean conjugate quaterni on obtained without complex conjugating v; in other words, if $q \in \mathbb{H}$ and $v \in \mathbb{C}$, then we define $\overline{vq} = v\bar{q}$.

where

$$\Omega = w_1\bar{w}_2 - w_2\bar{w}_1\,, \qquad L = |w_1|^2 + |w_2|^2\,,$$
$$\tau_1 = \frac{L}{H}\,, \qquad \tau_2 = \frac{L^2 - |\Omega|^2}{H^2}\,. \tag{5.22}$$

(5.21) reproduces the result known in literature [4].

With the aid of (5.16) we can now compute

$$\left\langle \frac{\mathrm{Tr}\,\phi^2}{8\pi^2} \right\rangle_{k=2} = -2\cdot(32\pi^2)^6 \int_{\mathcal{M}^+\setminus\{x_0\}} s^4 w_1 s^4 w_2 s^4 a_3 \det\widehat{h}|\det K| e^{-[S_B]_{k=2}}\,. \tag{5.23}$$

Using (4.27) and (5.21), it is easy to see that the integral over the bosonic moduli which appears in (5.23) is the same which was found in [5] after integrating out the fermionic zero-modes. As in [5], (5.23) thus leads to $\langle \mathrm{Tr}\,\phi^2/(8\pi^2)\rangle_{k=2} = -5\cdot(8\pi^2)^3/(4v^6)$ [5], which agrees with the results found by Seiberg and Witten in [1].

5.3 On the use of the operator s

As we have observed in the previous sections, the operator s is nilpotent. Moreover, it is possible to write the action $S_{\rm inst}$ as the operator s acting on a certain function of the moduli as in (4.37). This enables one to write the correlator $\langle \mathrm{Tr}\,\phi^2\rangle_k$ as an integral over the boundary of the instanton moduli space. Since

$$[S_{\rm inst}]_k = [S_B + S_F]_k = 4\pi^2 s\left\{\mathrm{Tr}\left[\bar{v}\left(\sum_{i=1}^{k}\mu_i\bar{w}_i - w_i\bar{\mu}_i\right)\right]\right\}\,, \tag{5.24}$$

and $s[S_{\rm inst}]_k = 0$, we obtain

$$\begin{aligned}
e^{-[S_{\rm inst}]_k} &= \sum_{l=0}^{\infty}\frac{(-)^l}{l!}\left([S_B]_k + [S_F]_k\right)\left([S_B]_k + [S_F]_k\right)^{l-1}\\
&= 4\pi^2 s\left\{\mathrm{Tr}\left[\bar{v}\left(\sum_{i=1}^{k}\mu_i\bar{w}_i - w_i\bar{\mu}_i\right)\right]\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\left([S_B]_k + [S_F]_k\right)^{l-1}\right\}\\
&= 4\pi^2 s\left\{\mathrm{Tr}\left[\bar{v}\left(\sum_{i=1}^{k}\mu_i\bar{w}_i - w_i\bar{\mu}_i\right)\right]\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\times\right.\\
&\qquad\left.\times\sum_{p=0}^{l-1}\binom{l-1}{p}\left([S_B]_k\right)^{l-1-p}\left([S_F]_k\right)^p\right\}.
\end{aligned} \tag{5.25}$$

Since $[S_F]_k$ is a fermion bilinear, in order to build a fermionic top form on the $(8k-4)$-dimensional reduced moduli space we must have $p = 4k-3$, leading to

$$
\begin{aligned}
e^{-[S_{\rm inst}]_k}\Big|_{\rm top\ form} &= 4\pi^2 s\left\{\mathrm{Tr}\left[\bar v\left(\sum_{i=1}^k \mu_i\bar w_i - w_i\bar\mu_i\right)\right]\frac{([S_F]_k)^{4k-3}}{(4k-3)!}\times\right.\\
&\qquad\left.\times\sum_{l=0}^{\infty}\frac{(-)^l}{(l+4k-2)}\frac{([S_B]_k)^l}{l!}\right\}\\
&= 4\pi^2 s\left\{\mathrm{Tr}\left[\bar v\left(\sum_{i=1}^k \mu_i\bar w_i - w_i\bar\mu_i\right)\right]([S_F]_k)^{4k-3}([S_B]_k)^{-4k+2}\times\right.\\
&\qquad\left.\times\left(1-e^{-[S_B]_k}\sum_{l=0}^{4k-3}\frac{([S_B]_k)^l}{l!}\right)\right\}. \qquad (5.26)
\end{aligned}
$$

As we stated in the introduction, writing the correlator as a total derivative over the moduli space can lead to interesting results. The $8k$-dimensional moduli space $\mathcal{M}_k$, can be compactified according to [34]. If we denote this compactification by $\overline{\mathcal{M}}_k$, it is well known that the boundary $\partial\overline{\mathcal{M}}_k$ can be decomposed into a union of lower moduli spaces, so that we can write

$$
\overline{\mathcal{M}}_k = \mathcal{M}_k \cup \mathbb{R}^4\times\mathcal{M}_{k-1}\cup S^2\mathbb{R}^4\times\mathcal{M}_{k-2}\cdots\cup S^k\mathbb{R}^4\,, \qquad (5.27)
$$

where $S^i\mathbb{R}^4$ denotes the ith symmetric product of points of $\mathbb{R}^4$. The curvature density in $S^l\mathbb{R}^4\times\mathcal{M}_{k-l}$ is

$$
|F_k|^2 = |F_{k-l}|^2 + \sum_{i=1}^l 8\pi^2\delta(x-y_i)\,, \qquad (5.28)
$$

where $y_i\in S^i\mathbb{R}^4$ are the centers of the instanton. We will check (5.28) in the $k=2$ case using (3.9). Given

$$
\begin{aligned}
&\Delta^\dagger\Delta = \qquad (5.29)\\
&= \begin{pmatrix} |w_1|^2+(x_1-x)^2+|a_1|^2 & \bar w_1 w_2+(\bar x_1-\bar x)a_1+\bar a_1(x_2-x)\\ \bar w_2 w_1+(\bar x_2-\bar x)a_1+\bar a_1(x_1-x) & |w_2|^2+(x_2-x)^2+|a_1|^2\end{pmatrix},
\end{aligned}
$$

we observe that one part of the boundary is given by $|w_1|\to 0$. Using (4.10) we have

$$
\lim_{|w_1|\to 0}\Delta \longrightarrow \begin{pmatrix} 0_{2\times 1} & \Delta_{k=1}\\ x_1-x & 0_{1\times 1}\end{pmatrix} \qquad (5.30)
$$

and

$$
\lim_{|w_1|\to 0}\det\Delta^\dagger\Delta = (x_1-x)^2\left[|w_2|^2+(x_2-x)^2\right]. \qquad (5.31)
$$

Then

$$
\begin{aligned}
\lim_{|w_1|\to 0}\mathrm{Tr}(FF)_{k=2} &= -\frac{1}{2}\lim_{|w_1|\to 0}\Box\Box\log\det(\Delta^\dagger\Delta)_{k=2}d^4x\\
&= -\frac{1}{2}\Box\Box\log(x_1-x)^2d^4x+\mathrm{Tr}(FF)_{k=1}\\
&= \mathrm{Tr}(FF)_{k=1}+8\pi^2\delta^4(x-x_1)\,. \qquad (5.32)
\end{aligned}
$$

Extending these computations to encompass all boundaries one can check (5.28). We leave the application of these considerations and of (5.26) to cases with $k > 1$ to future work and here we limit ourselves to a simple check of (5.26) in the $k = 1$ case.

From the analyses of [34, 41], it is known that the boundary of the $k = 1$ moduli space consists of instantons of zero "conformal" size; this means that if we projectively map the euclidean flat space $\mathbb{R}^4$ onto a four sphere S^4, the boundary of the corresponding transformed $k = 1$ instanton moduli space is given by instantons of zero conformal size τ, where τ is obtained from $|w|$ through a projective transformation ($|w|$ itself does not represent a globally defined coordinate on the S^4 instanton moduli space). In terms of the size $|w|$ of the $\mathbb{R}^4$ instanton, the $\tau \to 0$ limit corresponds to $|w| \to 0, \infty$. Specializing (5.26) to $k = 1$ and inserting it in (5.3) we get

$$\begin{aligned}\left\langle \frac{\mathrm{Tr}\,\phi^2}{8\pi^2} \right\rangle_{k=1} &= -\int_{\mathcal{M}^+\backslash\{x_0\}} e^{-[S_{\rm inst}]_{k=1}} = -\int_{\mathcal{M}^+\backslash\{x_0\}} e^{-4\pi^2\left[|v|^2|w|^2 - 2\mathrm{Tr}(\mu\bar{\mu}\bar{v})\right]} \\ &= -4\pi^2 \int_{\mathcal{M}^+\backslash\{x_0\}} s\Big\{ \mathrm{Tr}\left[\bar{v}(\mu\bar{w} - w\bar{\mu})\right] [S_F]_{k=1} \frac{1}{[S_B]^2_{k=1}} \times \\ &\qquad\qquad \times \left(1 - e^{-[S_B]_{k=1}} - [S_B]_{k=1} e^{-[S_B]_{k=1}}\right)\Big\} \\ &= 32\pi^4 \int_{\mathcal{M}^+\backslash\{x_0\}} s\Big\{ \mathrm{Tr}\left[\bar{v}(\mu\bar{w} - w\bar{\mu})\right] \mathrm{Tr}(\mu\bar{\mu}\bar{v}) \frac{1}{(4\pi^2|v|^2|w|^2)^2} \times \\ &\qquad\qquad \times \left(1 - e^{-4\pi^2|v|^2|w|^2} - 4\pi^2|v|^2|w|^2 e^{-4\pi^2|v|^2|w|^2}\right)\Big\}, \end{aligned} \tag{5.33}$$

where we have used (5.26) with

$$[S_B]_{k=1} = 4\pi^2|v|^2|w|^2\,, \tag{5.34}$$

$$[S_F]_{k=1} = -8\pi^2\,\mathrm{Tr}(\mu\bar{\mu}\bar{v})\,. \tag{5.35}$$

Using Stokes' theorem, we can compute $\langle \mathrm{Tr}\,\phi^2/(8\pi^2)\rangle_{k=1}$ as an integral over the boundary $\partial\left(\mathcal{M}^+\backslash\{x_0\}\right)$, which for $k = 1$ is $\partial\mathbb{R}^+ \times S^3/\mathbb{Z}_2$. Since

$$\begin{aligned} \mathrm{Tr}(\mu\bar{\mu}\bar{v}) &= -sw_\mu sw_\nu \eta^a_{\mu\nu}(v^a)^* = \left(\sigma^a_w s|w|^2 - |w|^2 s\sigma^a_w\right)(v^a)^*\,, \\ \mathrm{Tr}[\bar{v}(\mu\bar{w} - w\bar{\mu})] &= 2w_\mu sw_\nu \eta^a_{\mu\nu}(v^a)^* = 2|w|^2\sigma^a_w(v^a)^*\,, \end{aligned} \tag{5.36}$$

we get

$$\mathrm{Tr}(\mu\bar{\mu}\bar{v})\,\mathrm{Tr}[\bar{v}(\mu\bar{w} - w\bar{\mu})] = -4|w|^4(v^a v^a)^*\sigma^1_w\sigma^2_w\sigma^3_w\,. \tag{5.37}$$

Here σ^a_w are the left-invariant 1-forms, defined as $\sigma^a_w = |w|^{-2}\eta^a_{\mu\nu}w_\mu sw_\nu$, and satisfy the relation $s\sigma^a_w = \epsilon^{abc}\sigma^b_w\sigma^c_w$. Plugging (5.37) into (5.33) and recalling that

$\int_{S^3/\mathbb{Z}_2} \sigma_w^1 \sigma_w^2 \sigma_w^3 = \pi^2$, we get

$$\begin{aligned}\left\langle \frac{\operatorname{Tr}\phi^2}{8\pi^2} \right\rangle_{k=1} &= -\frac{8\pi^2}{v^2}\left(1 - e^{-4\pi^2|v|^2|w|^2} - 4\pi^2|v|^2|w|^2 e^{-4\pi^2|v|^2|w|^2}\right)\Bigg|_{|w|=0}^{|w|=\infty} \\ &= -\frac{8\pi^2}{v^2} \end{aligned} \tag{5.38}$$

which is the result obtained in (5.12).

6. Topological correlators in Witten's topological field theory

In this section we focus the attention on Witten's twisted formulation of $N = 2$ SYM theory. We will put to zero the v.e.v. v of the complex scalar field. For winding number $k = 1$, the top form on the (8-dimensional) instanton moduli space is $\operatorname{Tr}\phi^2(x_1)\operatorname{Tr}\phi^2(x_2)$, and one can compute the Green's function $\langle \operatorname{Tr}\phi^2(x_1)\operatorname{Tr}\phi^2(x_2)\rangle$. The prescription (4.1) for computing Green's function gives in this case

$$\big\langle \operatorname{Tr}\phi^2(x_1)\operatorname{Tr}\phi^2(x_2)\big\rangle = \int_{\mathcal{M}^+} \big[\operatorname{Tr}\phi^2(x_1)\operatorname{Tr}\phi^2(x_2)\big]_{\text{zero-mode subspace}}\,, \tag{6.1}$$

where we have recalled that the boundary term (2.50) in S_{TYM} vanishes when $v = 0$.

We could then proceed and compute explicitly the r.h.s. of (6.1). However, the observation that the BRST operator s is on $\mathcal{M}^+$ the exterior derivative leads us to consider, as in subsection 5.3, the possibility of computing correlators of s-exact operators as integrals of forms on the boundary of $\mathcal{M}^+$. Indeed, recall that we can write

$$\operatorname{Tr}\phi^2 = sK_c\,, \quad K_c = \operatorname{Tr}\left(csc + \frac{2}{3}ccc\right), \tag{6.2}$$

an expression which parallels the well-known relation

$$\operatorname{Tr}F^2 = dK_A\,, \quad K_A = \operatorname{Tr}\left(AdA + \frac{2}{3}AAA\right). \tag{6.3}$$

Using Stokes' theorem, one is led to re-express the r.h.s. of (6.1) as

$$\int_{\mathcal{M}^+} \operatorname{Tr}\phi^2 \operatorname{Tr}\phi^2 = \int_{\partial\mathcal{M}^+} K_c \operatorname{Tr}\phi^2\,. \tag{6.4}$$

We are then faced with two different computational strategies:

1. the bulk calculation, and

2. the boundary calculation.

Let us explore in detail both possibilities.

6.1 The calculation of $\langle \mathrm{Tr}\,\phi^2(x_1)\,\mathrm{Tr}\,\phi^2(x_2)\rangle$ in the bulk

From the last equation in (3.49) we know that the zero-mode configuration for ϕ is

$$\phi = U^\dagger \mathcal{M} f \mathcal{M}^\dagger U + U^\dagger \mathcal{A} U\,. \tag{6.5}$$

The parametrization for a $k = 1$ instanton has been described in section 4.1; from this it turns out that $f(x) = (\Delta^\dagger\Delta)^{-1} = [(x-x_0)^2 + w^2]^{-1}$. Plugging the expression (4.7) for $\mathcal{M}$ into (6.5) and recalling that $\mathcal{A} = 0$ when $k = 1$, we get

$$\phi = U^\dagger (s\Delta) f (s\Delta)^\dagger U\,. \tag{6.6}$$

It then follows that

$$\mathrm{Tr}\,\phi^2 = \mathrm{Tr}\left[Ps\Delta f(s\Delta)^\dagger Ps\Delta f(s\Delta)^\dagger\right], \tag{6.7}$$

where P has been introduced in (3.11). After a little algebra, (6.7) becomes

$$\mathrm{Tr}\,\phi^2(x) = -48 w^4 f^4(x) \prod_{\mu=1}^{4} \Gamma_\mu(x)\,, \tag{6.8}$$

where the quaternionic 1-form $\Gamma(x)$ is defined by

$$\Gamma(x) = s x_0 + \frac{(x-x_0)\bar{w}}{|w|^2} s w\,. \tag{6.9}$$

It is easy to convince oneself that

$$\prod_{\mu=1}^{4} \Gamma_\mu(x_1) \prod_{\nu=1}^{4} \Gamma_\nu(x_2) = J(x_1 - x_2) s^4 x_0 s^4 w\,, \tag{6.10}$$

with $J(x_1 - x_2) = (x_1 - x_2)^4/w^4$; we can then write

$$\mathrm{Tr}\,\phi^2(x_1)\,\mathrm{Tr}\,\phi^2(x_2) = (48)^2 w^4 (x_1 - x_2)^4 f^4(x_1) f^4(x_2) s^4 x_0 s^4 w\,. \tag{6.11}$$

Plugging this expression into the r.h.s. of (6.1), it follows that

$$\left\langle \mathrm{Tr}\,\phi^2(x_1)\,\mathrm{Tr}\,\phi^2(x_2)\right\rangle = (48)^2 (x_1 - x_2)^4 \int_{\mathcal{M}^+} s^4 x_0 s^4 w w^4 f^4(x_1) f^4(x_2)\,. \tag{6.12}$$

The structure of the $k = 1$ moduli space has been discussed in subsection 5.1, where we have learnt that $\mathcal{M}^+_{k=1} = \mathbb{R}^4 \times \mathbb{R}^+ \times S^3/\mathbb{Z}_2$. (6.12) then becomes

$$\int_{\mathbb{R}^+\times S^3/\mathbb{Z}_2} s^4 w w^4 \int_{\mathbb{R}^4} s^4 x_0 f^4(x_1) f^4(x_2) = \frac{\pi^4}{72} \frac{1}{(x_1 - x_2)^4}\,, \tag{6.13}$$

from which we finally get

$$\left\langle \frac{\mathrm{Tr}\,\phi^2(x_1)}{8\pi^2} \frac{\mathrm{Tr}\,\phi^2(x_2)}{8\pi^2} \right\rangle = \frac{1}{2}\,. \tag{6.14}$$

We remark that a hasty analysis would lead to the conclusion that, in the limit $|x_1 - x_2| \to 0$, the Green's function (6.14) is singular due to the behavior of (6.13). This is contrary to the geometrical interpretation of this correlator as a component of the Chern class of the bundle with curvature (2.11) [24, 23]. With a little more thinking one gets convinced that this singularity is only apparent, as we will show in the next section. In our opinion, this interpretation of the above-computed Green's function makes it unnatural the application to it of clustering arguments, as recently argued in [39].

We now turn to describe the same calculation performed on the boundary of instanton moduli space.[18]

6.2 The calculation of $\langle \mathrm{Tr}\,\phi^2(x_1)\,\mathrm{Tr}\,\phi^2(x_2)\rangle$ on the boundary of $\mathcal{M}^+$

We start off by considering (6.4), which allows us to write

$$\left\langle \mathrm{Tr}\,\phi^2(x_1)\,\mathrm{Tr}\,\phi^2(x_2)\right\rangle = \int_{\partial\mathcal{M}^+} K_c(x_1)\,\mathrm{Tr}\,\phi^2(x_2)\,. \tag{6.15}$$

The expression of the current K_c (which is a 3-form) is a trivial extension of (3.10), and reads, for instanton number k,

$$K_c = \mathrm{Tr}\left[PsD(sD)^\dagger D(sD)^\dagger + \frac{1}{3}\left(D^\dagger sD\right)\left(D^\dagger sD\right)\left(D^\dagger sD\right)\right]. \tag{6.16}$$

For $k = 1$ one simply has

$$D(x) = f^{1/2}(x)\begin{pmatrix} w \\ x_0 - x\end{pmatrix}, \tag{6.17}$$

and after a lengthy algebra, one finds

$$\begin{aligned} K_c(x) = 2f^3(x)\Big[&2w^4(w^2+3y^2)\bar\sigma^1_w\bar\sigma^2_w\bar\sigma^3_w + 2y^4(y^2+3w^2)\bar\sigma^1_y\bar\sigma^2_y\bar\sigma^3_y + \\ &+ 2w^2y^2(w^2+y^2)\left(\frac{sy^2}{2y^2} - \frac{sw^2}{2w^2}\right)\left(\bar\sigma^a_y\bar\sigma^a_w\right) + \\ &+ w^2y^2(y^2-w^2)s(\bar\sigma^a_y\bar\sigma^a_w)\Big], \end{aligned} \tag{6.18}$$

where we set $y = x_0 - x$. The right–invariant 1–forms $\bar\sigma^a_z$ are defined as $\bar\sigma^a_z = |z|^{-2}\bar\eta^a_{\mu\nu}z_\mu s z_\nu$, and satisfy the relation $s\bar\sigma^a_z = \epsilon^{abc}\bar\sigma^b_z\bar\sigma^c_z$. The next step consists in computing the product $K_c(x_1)\,\mathrm{Tr}\,\phi^2(x_2)$. The calculation is greatly simplified if one sets $x_1 = x_2$. Moreover, one has to take into account only the terms that yield a

[18] We thank Gian Carlo Rossi for many fruitful discussions and suggestions on the calculations described in the next section.

non-vanishing result when integrated on the boundary of instanton moduli space. If we do this, we get

$$\left[K_c \operatorname{Tr}\phi^2\right](x) \longrightarrow 192 y^4 w^4 f^4(x) \left(\bar{\sigma}^1_w \bar{\sigma}^2_w \bar{\sigma}^3_w\right) \left(\frac{s y^2}{2y^2} - \frac{s w^2}{2w^2}\right) \left(\bar{\sigma}^1_y \bar{\sigma}^2_y \bar{\sigma}^3_y\right). \tag{6.19}$$

Note that $y^4 (sy^2/2y^2)(\bar{\sigma}^1_y \bar{\sigma}^2_y \bar{\sigma}^3_y) = s^4(x_0 - x) = s^4 x_0$. It then follows that

$$\int_{\partial \mathcal{M}^+} \left[K_c \operatorname{Tr}\phi^2\right](x) = 192 \int_{S^3/\mathbb{Z}_2} \bar{\sigma}^1_w \bar{\sigma}^2_w \bar{\sigma}^3_w \lim_{|w|\to 0} \int_{\mathbb{R}^4} s^4 x_0 w^4 f^4(x). \tag{6.20}$$

Since

$$\begin{aligned}\lim_{|w|\to 0} \int_{\mathbb{R}^4} s^4 x_0 w^4 f^4(x) &= \lim_{|w|\to 0} \int_{\mathbb{R}^4} s^4 x_0 \frac{w^4}{[w^2 + (x - x_0)^2]^4} = C \int_{\mathbb{R}^4} s^4 x_0 \delta^{(4)}(x - x_0) \\ &= C\,, \end{aligned} \tag{6.21}$$

where

$$C = \int_{\mathbb{R}^4} s^4 x \frac{1}{(1 + x^2)^4} = \frac{\pi^2}{6}, \tag{6.22}$$

we conclude that

$$\int_{\partial \mathcal{M}^+} [K_c \operatorname{Tr}\phi^2](x) = 192 \pi^2 \cdot \frac{\pi^2}{6} = (8\pi^2)^2 \cdot \frac{1}{2}, \tag{6.23}$$

and the final result is

$$\left\langle \frac{\operatorname{Tr}\phi^2}{8\pi^2} \frac{\operatorname{Tr}\phi^2}{8\pi^2} \right\rangle = \int_{\partial \mathcal{M}^+} \frac{K_c}{8\pi^2} \frac{\operatorname{Tr}\phi^2}{8\pi^2} = \frac{1}{2}. \tag{6.24}$$

(6.24) coincides with the result found in (6.14). The limit of coincident points is thus well-defined, as we observed at the end of the previous section.

7. The ADHM construction and hyperkähler quotients

In this section we construct the moduli space of self-dual connections on flat space $\mathbb{R}^4$ in terms of hyperkähler quotients following [34]. This will allow us to clarify the geometrical meaning of the algebraic construction of the BRST transformations presented in section 3.4. As explained in [16], the hyperkähler construction of a quotient space can be regarded from a physicist's point of view as the gauging of a non-linear sigma model. The corresponding connection is obtained in a purely geometrical way *directly* from the isometries of the constraint equation (3.7) which imposes the self-duality of the gauge field strength (expressed in the ADHM formalism), and coincides with the connection $\mathcal{C}$ introduced in section 3.4 and worked out explicitly in section 4.1 for the $k = 2$ case. We will show that the square root of the determinant of the metric on instanton moduli space gives the bosonic jacobian [36] involved in the transformation of the functional integral into an integration over instanton moduli. For the construction of a gravitational instanton with this method see [40], while for an introduction to hyperkähler quotients in physicists' language see [16].

The starting point is the ADHM matrix a, which for the case of SU(2) instantons was defined in (3.2). Actually, for the present discussion it is more convenient to adopt a different parametrization for a; we rewrite it as

$$a = \begin{pmatrix} t & s^\dagger \\ A & -B^\dagger \\ B & A^\dagger \end{pmatrix}, \tag{7.1}$$

where A, B are $k \times k$ complex matrices and s, t are $k \times N$ and $N \times k$ dimensional matrices. Let us introduce the $4k^2 + 4kN$-dimensional hyperkähler manifold $M = \{A, B, s, t\}$. Given the three complex structures J^i_{ab} where $i = 1, 2, 3$ and $a, b = 1, \ldots, \mathrm{dim} M$, we can build the 2-forms $\omega^i = J^i_{ab} dx^a \wedge dx^b$, where x is a choice of coordinates on M. The real forms ω^i allow one to define a $(2,0)$ and a $(1,1)$ form

$$\begin{aligned} \omega_{\mathbb{C}} &= \mathrm{Tr}\; dA \wedge dB + \mathrm{Tr}\; ds \wedge dt\,, \\ \omega_{\mathbb{R}} &= \mathrm{Tr}\; dA \wedge dA^\dagger + \mathrm{Tr}\; dB \wedge dB^\dagger + \mathrm{Tr}\; ds \wedge ds^\dagger - \mathrm{Tr}\; dt^\dagger \wedge dt\,. \end{aligned} \tag{7.2}$$

The transformations

$$A \longrightarrow QAQ^\dagger\,, \qquad B \longrightarrow QBQ^\dagger\,, \qquad s \longrightarrow QsR^\dagger\,, \qquad t \longrightarrow RtQ^\dagger\,, \tag{7.3}$$

with $Q \in \mathrm{U}(k), R \in \mathrm{U}(N)$ leave $\omega_{\mathbb{C}}, \omega_{\mathbb{R}}$ invariant, and are the analogous of (3.12). Let ξ be a generator of the algebra which leaves ω^i invariant,

$$L_\xi \omega^i = 0\,, \tag{7.4}$$

where L_ξ is the Lie derivative along ξ. As ω^i is Kähler, (7.4) gives rise to conserved quantities, called momentum maps, defined as

$$i(\xi)\omega^i = d\mu^i_\xi\,, \tag{7.5}$$

where $\mu^i_\xi = \mu^i_a \xi^a$; in complex notation

$$\mu_{\mathbb{C}} = [A, B] + st\,, \qquad \mu_{\mathbb{R}} = \left[A, A^\dagger\right] + \left[B, B^\dagger\right] + ss^\dagger - t^\dagger t\,. \tag{7.6}$$

$\mu^i_\xi = 0$ defines a hypersurface

$$\mathcal{N}^+ = \Big\{\{A, B, s, t\} = x \in M:\; \mu^i_\xi = 0\Big\} \tag{7.7}$$

of dimension $\dim\mathcal{N}^+ = k^2 + 4kN$; using (7.1), one can immediately see that these equations are the equivalent of (3.15). The moduli space of self-dual gauge connections, $\mathcal{M}^+$, is obtained by modding $\mathcal{N}^+$ by the reparametrizations defined in (7.3). It has dimension $\dim\mathcal{M}^+ = 4kN$ and it is hyperkähler.[19]

In the following, we will focus on the $k = 2$ case with gauge group SU(2). For the explicit computations we go back to the parametrization of the ADHM moduli space introduced in section 3.1 and exploited for the $k = 2$ case in section 4.1; the matrix a is written in (4.8). We introduce a 20-dimensional hyperkähler manifold $M = (w_1, w_2, a_3, a_1, x_0)$.[20] Actually, since the theory is invariant under the group of translations in $\mathbb{R}^4$, one can fix x_0 and restrict the analysis to the 16-dimensional hyperkähler manifold $M\backslash\{x_0\}$ parametrized by the quaternionic coordinates $m^I = (w_1, w_2, a_3, a_1)$, endowed with a flat metric

$$ds^2 = \eta_{I\bar{J}} dm^I d\bar{m}^{\bar{J}} = |dw_1|^2 + |dw_2|^2 + |da_3|^2 + |da_1|^2 \,. \tag{7.8}$$

To keep the notation as simple as possible, we rename $M\backslash\{x_0\}$, $\mathcal{M}^+\backslash\{x_0\}$ and $\mathcal{N}^+\backslash\{x_0\}$ as M, $\mathcal{M}^+$, $\mathcal{N}^+$, respectively.

For $k = 2$, the ADHM bosonic constraint (3.15) reads

$$\bar{w}_2 w_1 - \bar{w}_1 w_2 = 2(\bar{a}_3 a_1 - \bar{a}_1 a_3) \,, \tag{7.9}$$

and, as discussed in section 3.1, it is invariant under the reparametrization group O(2), whose action on the $k = 2$ quaternionic coordinates is

$$\left(w_1^\theta, w_2^\theta\right) = (w_1, w_2) R_\theta \,, \qquad \left(a_3^\theta, a_1^\theta\right) = (a_3, a_1) R_{2\theta} \tag{7.10}$$

with

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{7.11}$$

The construction of the reduced bosonic moduli space $\mathcal{M}^+$ proceeds now in two steps. First, we solve explicitly (7.9) in an O(2) invariant way. Since the constraint (3.7) corresponds to $3k(k-1)/2$ equations, $\mathcal{N}^+$ turns out to be a 13-dimensional manifold for $k = 2$, described by the set of coordinates (w_1, w_2, a_3, Σ), where Σ is the auxiliary real variable related to the O(2) reparametrization symmetry. Second, we mod out this isometry group of $\mathcal{N}^+$ by means of the hyperkähler quotient procedure. The instanton moduli space is then $\mathcal{M}^+ = \mathcal{N}^+/\,\mathrm{O}(2)$, and it has dimension $\dim\mathcal{M}^+ =$

[19] The metric on $\mathcal{M}^+$ could also be obtained from the Kähler form $\omega_{\mathcal{M}^+}$, which in turn is expressed in terms of the Kähler potential $\mathcal{K}$, as [37] $\omega_{\mathcal{M}^+} = \partial\bar{\partial}\mathcal{K} = \frac{1}{2}\partial\bar{\partial}\,\mathrm{Tr}\left[a^\dagger(1 + P_\infty)a\right]$, where $P_\infty = 1 - bb^\dagger$ is the asymptotic expression of the projector P defined in (3.11).

[20] Notice that, since we are using a different parametrization of the ADHM space with respect to (7.1), the dimension of the manifolds M and $\mathcal{N}^+$ is not that of the previous discussion. However, also the reparametrization groups are different, in such a way that the final dimension of the moduli space of self-dual gauge connections is the same, as it must be.

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$\dim\mathcal{N}^+ - k(k-1)/2|_{k=2} = 12$. As anticipated, the construction of the quotient space $\mathcal{M}^+$ can be seen as the gauging of a non-linear sigma model. The corresponding connection is given by [16]

$$\mathcal{C} = \frac{1}{2|k|^2}\eta_{I\bar{J}}\left(\bar{k}^{\bar{J}}dm^I + d\bar{m}^{\bar{J}}k^I\right), \tag{7.12}$$

where $k^I\partial_I + \bar{k}^{\bar{I}}\bar{\partial}_{\bar{I}}$ is the $\mathrm{O}(k)$ Killing vector with $|k|^2 = \eta_{I\bar{J}}k^I\bar{k}^{\bar{J}}$. The components of the $\mathrm{O}(2)$ Killing vector on M are easily deduced from (7.10):

$$k^I = (-w_2, w_1, -2a_1, 2a_3)\,. \tag{7.13}$$

Substituting (7.13) into (7.12), we get

$$\begin{aligned}\mathcal{C} = \frac{1}{2H}\Big(&\bar{w}_1dw_2 - \bar{w}_2dw_1 + 2\bar{a}_3da_1 - 2\bar{a}_1da_3 + \\ &+ d\bar{w}_2w_1 - d\bar{w}_1w_2 + 2d\bar{a}_1a_3 - 2d\bar{a}_3a_1\Big)\,.\end{aligned} \tag{7.14}$$

Notice that this is exactly the connection (4.16) obtained in section 4.1 by solving the fermionic constraint (3.19). Therefore, this procedure clarifies the geometrical meaning of the connection $\mathcal{C}$ introduced in section 3.1, providing a very simple method to compute it *directly* from the isometries of the ADHM moduli space, without referring to the constraint equation (3.19).

The metric $g^{\mathcal{N}^+}_{I\bar{J}}$ on the constrained hypersurface $\mathcal{N}^+$ is obtained plugging (4.10) into (7.8), and gets simplified if we introduce the variable

$$W = \bar{w}_2w_1\,. \tag{7.15}$$

The hypersurface $\mathcal{N}^+$ is now described by the new set of coordinates (w_1, U, V, a_3, Σ), where

$$U = \frac{W+\overline{W}}{2}\,, \qquad V = \frac{W-\overline{W}}{2}\,, \tag{7.16}$$

are respectively the real and the imaginary part of W. The jacobian factor associated to this change of variables is

$$d^4w_1dUd^3V = |w_1|^4d^4w_1d^4w_2\,. \tag{7.17}$$

In the new variables, (7.8) reads

$$\begin{aligned}ds^2 = &\left(1+\frac{|w_2|^2}{|w_1|^2}\right)|dw_1|^2 + \frac{dU^2}{|w_1|^2} + \frac{|dV|^2}{|w_1|^2} - \frac{dU}{|w_1|^2}(\bar{w}_2dw_1 + d\bar{w}_1w_2) + \\ &+ \frac{dV}{|w_1|^2}(\bar{w}_2dw_1 - d\bar{w}_1w_2) + |da_3|^2 + |da_1|^2\,,\end{aligned} \tag{7.18}$$

which, inserting (4.10), becomes

$$\begin{aligned} ds^2 = & \left(1+\frac{|w_2|^2}{|w_1|^2}\right)|dw_1|^2+\frac{dU^2}{|w_1|^2}+\frac{|dV|^2}{|w_1|^2}-\frac{dU}{|w_1|^2}(\bar{w}_2 dw_1+d\bar{w}_1 w_2)+ \\ & +\frac{dV}{|w_1|^2}(\bar{w}_2 dw_1-d\bar{w}_1 w_2)+\left(1+\frac{|a_1|^2}{|a_3|^2}\right)|da_3|^2+\frac{d\Sigma^2}{16|a_3|^2}+\frac{|dV|^2}{4|a_3|^2}- \\ & -\frac{d\Sigma}{4|a_3|^2}(\bar{a}_1 da_3+d\bar{a}_3 a_1)-\frac{dV}{2|a_3|^2}(\bar{a}_1 da_3-d\bar{a}_3 a_1)\,. \end{aligned} \tag{7.19}$$

The r.h.s. of (7.19) can be regarded as the lagrangian density of a zero-dimensional non-linear sigma model with target space $\mathcal{N}^+$. In real coordinates $m^A=(w_1^\mu, U, V^i, a_3^\mu, \Sigma)$, the $\mathrm{O}(2)$ Killing vector on this manifold has components

$$k^A=\left(-w_2^\mu, |w_1|^2-|w_2|^2, 0, -2a_1^\mu, 8\left(|a_3|^2-|a_1|^2\right)\right). \tag{7.20}$$

The global $\mathrm{O}(2)$ symmetry can be promoted to a local one by introducing the connection (7.12), which on $\mathcal{N}^+$ is written as

$$\mathcal{C}=\frac{g^{\mathcal{N}^+}_{AB}k^B}{H}dm^A=\frac{1}{H}\left(-2w_2^\mu dw_1^\mu+dU-4a_1^\mu da_3^\mu+\frac{d\Sigma}{2}\right), \tag{7.21}$$

where the metric $g^{\mathcal{N}^+}_{AB}$ is obtained by rewriting (7.19) in the coordinates $\{m_A\}$. Writing U in terms of w_1, w_2 by means of (7.15) and (7.16), the connection (7.21) becomes

$$\mathcal{C}=\frac{1}{H}\left(w_1^\mu dw_2^\mu-w_2^\mu dw_1^\mu-4a_1^\mu da_3^\mu+\frac{d\Sigma}{2}\right). \tag{7.22}$$

From the gauged version of the lagrangian (7.19) we can read off the metric on $\mathcal{M}^+=\mathcal{N}^+/\,\mathrm{O}(2)$ written in the $\{m^A\}$ coordinates, namely [16]

$$g^{\mathcal{M}^+}_{AB}=g^{\mathcal{N}^+}_{AB}-\frac{g^{\mathcal{N}^+}_{AC}\,g^{\mathcal{N}^+}_{BD}\,k^C\,k^D}{g^{\mathcal{N}^+}_{EF}\,k^E\,k^F}\,. \tag{7.23}$$

The local $\mathrm{O}(2)$ isometry allows one to put Σ to zero; notice that in this gauge (7.22) leads to the connection (4.17). Finally, by using translational invariance to restore the dependence on x_0, and taking into account the jacobian factor (7.17), we write the volume form on the moduli space of self-dual gauge connections with winding number $k=2$ as

$$|w_1|^4\sqrt{g^{\mathcal{M}^+}_{\Sigma=0}}\,d^4w_1 d^4w_2 d^4a_3 d^4x_0=\frac{H}{|a_3|^4}\Big||a_3|^2-|a_1|^2\Big|\,d^4w_1 d^4w_2 d^4a_3 d^4x_0\,, \tag{7.24}$$

which reproduces the well-known result of Osborn [38].

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